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BULLETIN

OF THE

AMERICAN
MATHEMATICAL SOCIETY.

CONTINUATION OF THE BULLETIN OF THE NEW YORK MATHEMATICAL SOCIETY.

A HISTORICAL AND CRITICAL REVIEW
OF MATHEMATICAL SCIENCE.

EDITED BY
THOMAS S. FISKE, ALEXANDER ZIWET, FRANK MORLEY.

2ND SERIES, VOL. II. No. 1.

OCTOBER, 1895.

NEW YORK:
PUBLISHED FOR THE SOCIETY
BY MACMILLAN & CO.
1895.

Entered at the Post-office at New York, N. Y., as second-class matter.
PUBLISHED MONTHLY, EXCEPT AUGUST AND SEPTEMBER.
Whole No. 41. $5.00 a yea

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BULLETIN OF THE AMERICAN MATHEMATICAL SOCIETY.

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VOL. II.

October 1895 to July 1896.

NEW YORK:
PUBLISHED FOR THE SOCIETY
BY
THE MACMILLAN COMPANY.
1896.

THE NEW ERA PRINT,
LANCASTER, PA.

63441

BULLETIN OF THE

AMERICAN MATHEMATICAL SOCIETY

THE SECOND SUMMER MEETING OF THE AMERICAN MATHEMATICAL SOCIETY.

THE Second Summer Meeting of the American Mathematical Society was held in the High School building at Springfield, Mass., on August 27 and 28. The society enjoyed the co-operation of the American Association for the Advancement of Science, the regular sessions of which began on August 29, and the members of the society were entitled to all the privileges extended to the members of the latter organization. Among those present were: Prof. A. L. Baker, Dr. J. H. Boyd, Prof. C. H. Chandler, Prof. L. L. Conant, Prof. C. L. Doolittle, Prof. W. P. Durfee, Prof. E. Frisby, Dr. G. D. Gable, Dr. J. W. L. Glaisher, Prof. A. Hall, Jr., Prof. Ellen Hayes, Dr. G. W. Hill, Prof. W. Woolsey Johnson, Mr. P. A. Lambert, Prof. J. McMahon, Prof. M. Merriman, Prof. F. Morley, Prof. H. B. Newson, Prof. W. F. Osgood, Prof. M. I. Pupin, Mr. R. A. Roberts, Mr. C. H. Rockwell, Prof. J. B. Shaw, Prof. W. E. Story, Prof. H. Taber, Prof. J. M. Van Vleck, Prof. E. B. Van Vleck, Prof. C. A. Waldo, Prof. H. S. White, Prof. J. M. Willard, Prof. C. B. Williams, Prof. F. S. Woods, and Prof. R. S. Woodward.

The President, Dr. G. W. Hill, occupied the chair. Two sessions were held each day, meeting respectively at 10 A.M. and 2.30 P.M. On the recommendation of the Council, Dr. Thomas Jefferson J. See, University of Chicago, Chicago, Ill., was elected to membership. Three nominations for membership were received.

The following papers were read:

1. *The periodic solution as a first approximation in the lunar theory.* Dr. G. W. HILL.
2. *The linear vector operator of quaternions.* Professor J. B. SHAW.
3. *A new application of quaternions to geometry.* Professor J. B. SHAW.

4. *On a generalization of Weierstrass's equation with three terms.* Professor F. MORLEY.

5. *Formulas for the sides of rational plane triangles.* Dr. ARTEMAS MARTIN.

6. *Partial linear transformations of ternary quantics and their concomitants.* Professor J. McMAHON.

7. *An introduction to the integrell calculus.* Professor W. H. ECHOLS.

8. *On the expansion of a uniform function of a real variable without use of derivatives.* Professor W. H. ECHOLS.

9. *On continuous functions without differential coefficients.* Mr. P. A. LAMBERT.

10. *Concerning Jordan's linear substitution groups.* Professor E. H. MOORE.

11. *Algebraic symbols and* $\sqrt{-1}$. Professor A. L. BAKER.

12. *An application of the method of conformal representation to the study of related differential equations.* Professor E. B. VAN VLECK.

13. *On the differential equations of certain systems of conics.* Mr. R. A. ROBERTS.

14. *On bilinear forms.* Professor H. TABER.

15. *Elementary proof of the quaternion associative principle.* Professor A. S. HATHAWAY.

16. *Asymptotic lines on a circular ring.* Professor H. MASCHKE.

Dr. Hill's paper, since its presentation, has been published in the *Astronomical Journal,* No. 353, September 7, 1895, pp. 137–143. Its object is to obtain the values of the coefficients of the periodic inequalities having the multiples of the mean angular distance of the Moon from the Sun as arguments, when the inclination of the lunar orbit and the two eccentricities are neglected. It is very desirable to have these coefficients with a high degree of accuracy in order to effect their useful employment in the further determination of the motion of the perigee and node, and in fact of all the other coefficients of the periodic inequalities. This work has been done previously by the author in the *American Journal of Mathematics,* vol. 1, but in neglecting the lunar mass and the solar parallax. Mr. Ernest W. Brown has in the same journal supplemented these researches, but still leaving out of consideration the mass of the Moon.

Professor Shaw's first paper is a development of the linear vector operator of quaternions in the form of a scalar part and two vector parts, also as a tensor and a versor, and finally as a sum of nine operators. The forms are similar to those obtained in the articles of Professor Taber (*Amer. Jour. Math.,* vols. 12, 13), but are here developed entirely from quaternion expressions and not from matrices. It was stated that this paper would be offered to the *American Journal of Mathematics* for publication.

Professor Shaw's second paper applies the quaternion calculus to homogeneous geometry of two dimensions. In the expression $\zeta = xi + yj + zk$, x, y, z are proportional to the areas PBC, PCA, PAB respectively where the point P is referred to the fundamental triangle ABC. By this convention propositions of projective geometry are easily proved, and especially such propositions of modern geometry as those of the Lemoine-Brocard type. The author expected to contribute this paper to the *Annals of Mathematics*.

Professor Morley's paper is published in the present number of the BULLETIN. It contains a simple generalization of the formula

$$\begin{aligned} &\sigma(u + u_1)\ \sigma(u - u_1)\ \sigma(u_2 + u_3)\ \sigma(u_2 - u_3) \\ +\ &\sigma(u + u_2)\ \sigma(u - u_2)\ \sigma(u_3 + u_1)\ \sigma(u_3 - u_1) \\ +\ &\sigma(u + u_3)\ \sigma(u - u_3)\ \sigma(u_1 + u_2)\ \sigma(u_1 - u_2) = 0. \end{aligned}$$

In Dr. Martin's paper a large number of formulæ are deduced for calculating rational numbers which represent the sides of triangles having a rational area. Among them are, for the case of a right-angled triangle,

$$x = 2pq, \qquad y = p^2 - q^2, \qquad z = p^2 + q^2,$$

and, for the case of an oblique triangle,

$$\begin{aligned} x &= (p^2 + q^2)(r^2 - s^2), \\ y &= 2rs(p^2 + q^2) \pm 2rs^2(p^2 - q^2), \\ z &= (p^2 + q^2)(r^2 + s^2) \pm 2rs(p^2 - q^2), \end{aligned}$$

in which x, y, z denote the sides, and p, q, r, s any entire numbers whatever. The paper, which contains many numerical applications, will be printed in the *Mathematical Magazine*. In the absence of Dr. Martin his paper was read by the Secretary.

Semi-invariants of a ternary quantic satisfy some, but not all, of the six differential equations which characterize invariants. Professor McMahon's paper shows how to distinguish between those semi-invariants which are the sources of covariants and those which are the sources of semi-covariants, and gives a simple method of deriving from the latter all the coefficients of the semi-covariant, or of a semi-contravariant if desired. A systematic geometrical interpretation of these three kinds of semi-concomitants is presented and appears to be particularly useful in Cartesian co-ordinates. This paper is intended for the *Annals of Mathematics*.

In the absence of Professor Echols his two papers were presented by the Secretary. The first one was the second section of an essay "On the differell and integrell calculus," the preceding section having been read at the May meeting of the Society. Professor Echols approaches the infinitesimal calculus from the calculus of finite differences. He considers the latter, however, in a greatly generalized form. A differell is defined to be the limit of a ratio whose terms are the nth differences of the function and the independent variable. An integrell is a differell of negative index. The applications presented consisted chiefly in the expansion of functions in terms of differells and integrells. In the second paper some of the more novel results were translated into the language of ordinary calculus.

Mr. Lambert attempted to show that functions of which Weierstrass's derivativeless function is the type may be so considered geometrically that their curves will have determinate tangents. In order to secure agreement of the analytic result with the geometric he found it necessary to replace Weierstrass's supposition in regard to h

$$\tfrac{1}{2}\frac{\pi}{a^n} < h < \tfrac{3}{2}\frac{\pi}{a^n},$$

by the limitation

$$0 < h < \epsilon\frac{\pi}{a^{2n}},$$

n increasing indefinitely, and ϵ being any finite quantity.

In Professor Moore's paper a tactical configuration is established which exactly defines Jordan's linear substitution group when taken fractionally. This configuration is self-reciprocal. The properties of this configuration and of other allied configurations similarly related to certain important subgroups of the main group are developed. Professor Moore's paper will appear in the BULLETIN OF THE AMERICAN MATHEMATICAL SOCIETY. It was presented at the meeting by the Secretary.

Professor Baker's paper contained a detailed discussion of the character of the operations of algebra and the theory of imaginary quantities. It will be published in the *American Journal of Mathematics*.

In Professor Van Vleck's paper an aggregate of regular linear differential equations of the second order is taken, each of which has a polynomial solution. The requirement is made that these equations shall have a common group arising from their four common branch points, or in Riemann's phraseology, that they shall be related. This requirement necessitates the introduction of one accessory branch point into each equation. These accessory points do not, however, give rise to any substitutions of the group. The paper outlines a method by which the position of the accessory points may be investigated, as well as the distribution of the roots of the various polynomials between the four (real) branch points and in the imaginary domain. One result is the determination of the distribution of the roots of all polynomials satisfying differential equations of the second order with exactly four branch points which with their exponent differences are given.

Mr. Roberts' paper gives the differential equations of certain systems of conics in a plane, viz., conics having double contact with two fixed conics, etc., and in space, of conics touching six fixed planes, conics having double contact with three quadrics inscribed in the same developable, circles having double contact with two confocal quadrics, etc. These results are principally deduced by means of elliptic integrals and the first-class of hyper-elliptic integrals, and from them flow certain theorems concerning doubly infinite porisms of curvilinear polygons. The paper appears in the present number of the BULLETIN.

Professor Taber's paper contains a theorem concerning the transformation into itself by an orthogonal substitution of an alternate bilinear form with cogredient variables, and treats of the consequences of this theorem. Thus, denoting by G the group of linear substitutions which transforms into itself an alternate bilinear form with cogredient variables, the theorem in question is that every orthogonal substitution of the group G is the second power of a substitution of this group. From this theorem may be derived: (*a*) The conditions necessary and sufficient that a given substitution of the group G may be generated by the repetition of an infinitesimal substitution of the group. (*b*) A proof that every substitution of the group G can be obtained by the composition of three linear substitutions given by Cayley's rational representation for the substitutions of this group. (*c*) A representation of the coefficients of any orthogonal substitution of the group G as rational functions of the minimum number of parameters, provided the alternate bilinear form is orthogonal. Further it follows from the theorem that every linear substitution which transforms into itself a general bilinear form F is the second power of such a substitution, and can be generated by the repetition of an infinitesimal substitution which also transforms F into itself, provided, if F' denotes the form conjugate to F, that the forms $F + F'$ and $F - F'$ are both of non-evanescent determinant. The paper also contains a solution of the problem to determine the linear transformations between two alternate bilinear forms, depending upon an algebraic equation of degree equal to the number of variables of the forms. The paper will be contributed to the *Quarterly Journal of Pure and Applied Mathematics.*

In Professor Hathaway's paper the proof consists in identifying, by means of elementary geometry, the product of several versors with the composition of a set of rotations through angles double those of the corresponding versors. The obvious associative principle of the composition of the rotations proves the corresponding associative principle of multiplication of versors. This paper will appear in the BULLETIN OF THE AMERICAN MATHEMATICAL SOCIETY. It was presented to the Society by Professor Shaw.

Professor Maschke's paper contains a very elegant application of elliptic functions to curves drawn on the surface of a circular ring. This paper appears in the present number of the BULLETIN. It was presented to the Society by Professor Morley.

At the afternoon session, August 28th, two topics were presented to the Society for general discussion:

(1) "A general subject catalogue or index of mathematical literature."

(2) "The mathematical curriculum of the college and scientific school."

The first discussion was opened by the Secretary, who gave a brief account of the "*Répertoire bibliographique des sciences mathématiques,*" in course of publication by the Mathematical Society of France. The discussion was continued by Professors Morley, Woodward, and McMahon. On motion by Professor McMahon, it was resolved that the Council be requested to consider the desirability of offering to the Mathematical Society of France the coöperation of this Society and of drawing up a plan for such coöperation.

The second discussion was opened by Professor Shaw, who presented a table of statistics showing the character of the mathematical instruction in 101 representative colleges and scientific schools. The discussion was continued by Professors White, Morley, Van Vleck, Doolittle, Chandler, Pupin, and Woodward. It seemed to be generally held that the work of the preparatory schools as a whole is not sufficiently thorough to serve as a satisfactory basis for collegiate courses, that a greater proportion of the students' time should be given to mathematical study, that greater stress should be laid on the fundamental subjects, that elementary portions of applied mathematics should be earlier introduced and more extensively taught, and that spherical trigonometry should be in great part, or altogether, dropped from the required curriculum.

At the close of the discussion the thanks of the Society were tendered to the Springfield Local Committee for the accommodations and hospitality which the Society had enjoyed, and the meeting was adjourned.

THOMAS S. FISKE.

COLUMBIA COLLEGE, *September* 10, 1895.

THE MEETING OF THE AMERICAN ASSOCIATION.

THE forty-fourth annual meeting of the American Association for the Advancement of Science was held at Springfield, Mass., August 29–September 4, President E. W. Morley, of Cleveland, Ohio, presiding. The attendance was not large, only 367 members being registered. This is 110 less than the number attending the Brooklyn meeting last year. There were 113 new members and 58 fellows elected this year.

Excursions were given to Forest Park, Springfield; to Amherst, Northampton, and Mount Holyoke College, and to Holyoke.

The officers of Section A were as follows: Edgar Frisby, Washington, D. C., Vice-President; Asaph Hall, Jr., Ann Arbor, Mich., Secretary; C. A. Waldo, Greencastle, Ind., Councilor; J. E. Kershner, Lancaster City, Pa.; C. A. Doolittle, Philadelphia, Pa.; L. L. Conant, Worcester, Mass.; C. H. Rockwell, Tarrytown, N. Y., together with the Vice-President and Secretary, sectional committee; J. R. Eastman, Washington, D. C.; S. C. Chandler, Cambridge, Mass.; J. M. Van Vleck, Middletown, Conn., with the Vice-President and Secretary, committee to nominate officers of the section. A. N. Skinner, D. C., Press Secretary.

The work of the Section was devoted almost wholly to astronomy, only two papers in pure mathematics being presented. The tendency of the Section to devote itself to the former class of work was commented on and deplored, and it was hoped that this tendency might be arrested before it had gone too far, and the balance between mathematical and astronomical work restored at the next meeting. This feeling manifested itself to a certain extent in the choice of officers, for the ensuing year, mentioned below.

The papers presented to Section A were as follows:

1. *Development of some useful quaternion expressions, with applications to geometry of three and four dimensions.* By JAMES BYRNIE SHAW, Jacksonville, Ill.
2. *The constant of aberration.* By C. L. DOOLITTLE, Philadelphia, Pa.
3. *On the constant of nutation.* By S. C. CHANDLER, Cambridge, Mass.

4. *Progress of the zone-work at the Naval Observatory, Washington, D. C.* By A. N. Skinner, Washington, D. C.
5. *Sun-spots and magnetic storms.* By M. A. Veeder, Lyons, N. Y.
6. *Notes on square numbers whose sum is either a square or the sum of other squares.* By Artemas Martin, Washington, D. C. Read by the Secretary.
7. *A convenient formula for computing times of moon rising.* By Edgar Frisby, Washington, D. C.
8. *On a slide scale for computing precession.* By Edgar Frisby, Washington, D. C.
9. *Some results for stellar parallax from meridian transit observations at the Washburn Observatory.* By A. S. Flint, Madison, Wis.
10. *The spectrum of β Lyræ.* By E. B. Frost, Hanover, N. H. Read by the Secretary.
11. *Chronology and ancient eclipses.* By S. W. Baloh, Yonkers, N. Y.
12. *Period of R Comæ.* By H. M. Parkhurst, Brooklyn, N. Y.
13. *On the distribution and the secular variation of terrestrial magnetism.* By L. A. Bauer, Washington, D. C. By title.
14. *On making astronomy popular.* By Miss Mary Procter. By title.

No. 1 was a development of the A-process in quaternions, where A is defined by the equation $A\,.\,pq = \frac{1}{2}(pq - qp)$. This paper developed the four alternating functions

$$\begin{aligned} &A\,.\,pq = \tfrac{1}{2}(pq - qp),\\ &S\,.\,pAqr,\\ &A\,.\,pqr = S\,.\,pAqr - Sp\,.\,A\,.\,qr - Sq\,.\,Arp - Sr\,.\,Apq,\\ &S\,.\,pAqrs. \end{aligned}$$

A set of four quaternions related to one another is deduced, analogous to a set of three rectangular unit-vectors. Various collections of formulas are deduced.

Affixing one of the set of four quaternions to each vertex of a tetrahedron, and letting $P = x_1 l_1 + x_2 l_2 + x_3 l_3 + x_4 l_4$ be that point for which the volumes $PABC$, $PBCD$, $PCDA$, $PDAB$ are as $x_1 : x_2 : x_3 : x_4$, we are enabled to treat solid geometry projectively.

No. 2 was a description of the method employed by Professor Doolittle in determining the constant of aberration. For this purpose a series of 2796 zenith telescope observations were made during the interval from Oct. 10, 1892, to Dec. 27, 1893. Of these observations, 1744 were made before midnight, and 1052 after midnight. Each observation gives an equation of the form

$$\phi_0 + k + Ax + By + Cz + Du + E\rho + \tau\mu = \phi$$

where x, y, z, u depend upon the two terms of the latitude variation, ρ is the correction to the assumed value of the constant of aberration, and μ the secular change of the latitude. These equations were combined so as to eliminate ϕ_0 and k, leaving 6 unknown quantities. A least-square solution gives for the constant of aberation 20″.552 ± .009.

No. 3 consisted of (*a*) a new determination of the constant of lunar nutation from an extensive body of observations, not hitherto employed for this purpose, and calculated by its distribution to avoid constant sources of error affecting many previous determinations; (*b*) a combination of this value with those deduced by previous investigations, to determine the most probable value of this fundamental constant of astronomy.

No. 4 detailed the relation of that part of the work at the Naval Observatory at Washington under Mr. Skinner's personal direction, to the general scheme of the zone-work planned by the German Astronomical Society. A description was given of the methods employed and the instruments used in this work, and mention was made of the difficulties encountered and of the methods of overcoming them as the work progressed.

No. 5 exhibited printed tables, not heretofore published, illustrating methods of recording sun-spots and magnetic phenomena so as to bring out their relations to each other, and the periodicities involved. Dr. Veeder believed that the records of magnetic observatories might be put in form so as to show the extent of the daily disturbances in such a manner that they could be printed and made readily accessible to the public.

No. 6 developed theorems relating to

(1) *n* square numbers whose sum is a square.

(2) Three square numbers the sum of whose squares is a square.

(3) Five square numbers the sum of whose squares is a square.

(4) *n* square numbers the sum of whose squares is a square.

(5) Three square numbers whose sum is equal to the sum of two other squares.

(6) Four square numbers whose sum is equal to the sum of two other squares.

(7) Four square numbers whose sum is equal to the sum of three other squares.

(8) m square numbers whose sum is equal to the sum of n other squares.

(9) $S(1^2) + S(2^2) + S(3^2) + S(4^2) + \ldots\ldots + S(n^2) = \square$, where $S(x^2) = 1^2 + 2^2 + 3^2 + \ldots\ldots + x^2$.

(10) Three square numbers, the sum of every two of which is a square.

(11) Four square numbers, the sum of every three of which is a square.

This paper will be published in the *Mathematical Magazine.*

No. 9 detailed the method of observation used at the Washburn Observatory, which is the same as that employed at the Leyden Observatory, 1885–87, by Prof. J. C. Kapteyn. It consists in observing the differences in the time of meridian transit between the parallax star and two companion stars, one preceding and one following. A list of 76 stars was observed, the Repsold meridian circle being used. These observations included all stars known to have a proper motion of 1″ or more; and some rapid binaries and stars of the first and second magnitude were added later. The results obtained agree with those obtained by previous observers. This paper is to be published in the *Astronomical Journal.*

No. 10 gave a brief account of some of the problems regarding the spectrum of β Lyræ, which have been brought to light by the recent spectroscopic observations of this star at Pulkowa, Potsdam, and London.

No. 11 contained some original researches relating to the bearing on astronomical computation of certain irregularities in the Roman calendar. In astronomical calculations which extend back to early periods of history, it is customary to follow the Old Style calendar. When astronomical events thus computed are also dated in the historical record it is found that the record and computation agree as to the day of the month, provided the date is subsequent to March 1 of 1 B.C. Prior to that date the Roman calendar was more or less irregular, but it has not been customary to follow these irregularities in astronomical work. Instead, the Old Style calendar has been extended backward as far as necessary for calculation; and is, to that extent, fictitious. The actual calendar as observed by the Romans differs from the fictitious or astronomical calendar by amounts varying according to the time. It is important in historical comparisons to know what this difference is; and this was shown in tables prepared for the purpose by Mr. Balch. History indicates that Cæsar began the calendar with a new moon. Computation shows that in 46 B.C. the moon was new on January 2 of the Old Style calendar. Mr. Balch's table showed that the kalends of January as observed coincided with the day preceding January 1 of 46 B.C. The new moon occurred, therefore, two days after the calendar began.

Hence Cæsar's astronomer, Sosogenes, was so much out in his reckoning; but the people doubtless did not observe it, supposing that the new undertaking was begun with the new moon, in accordance with their superstitions.

No. 12 gave the results of observations from 1831 to 1895, on R Comæ, from which it is evident that the period of this star, which was formerly 361 days, has become 365 days; and that, if the period continues to lengthen, the maxima will soon cease to be observable. If, however, we adopt a sine formula, equally well representing all the observations, we find that the period will shorten, bringing the maximum periods out of the twilight and rendering them observable. But if the period should continue to lengthen, the maximum will not be visible again for many years. Either of these theories accords perfectly well with recorded observations; but the former, depending as it does on the sine formula, seems now the more probable.

The next annual meeting of the Association will be held at Buffalo. Instead of beginning in the middle of the week, as has been the custom during the past few years, the sessions will begin on Monday and continue until Friday night. No excursions are to be given until after the close of the meeting. The President for the ensuing year will be E. D. Cope, of Philadelphia.

The officers of Section A for 1896 are as follows: Vice-President, W. E. Story, of Worcester, Mass., and Secretary, E. B. Frost, Hanover, N. H. LEVI L. CONANT.

WORCESTER, MASS., *September* 14, 1895.

ON THE DIFFERENTIAL EQUATIONS OF CERTAIN SYSTEMS OF CONICS.*

BY MR. R. A. ROBERTS.

IN a few cases certain systems of curves can be represented very simply by differential equations of the first order. The latter, since they do not involve the parameter, give readily the tangents to curves of the system, at an arbitrary point of space or of the surface to which the curves are confined. Hence certain loci connected with the system and various properties of the curves can be obtained without having recourse to the integral. A remarkable example of this is to be found in the case of the geodesics on a quadric, all whose properties are obtained by means of the differential equation of the first order, without having recourse to the transcendental final integral.

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* **Read before the AMERICAN MATHEMATICAL SOCIETY at its Second Summer Meeting, Springfield, Mass., August 28, 1895.**

A short review of the kind of differential equations that I propose to investigate may be of interest.

Considering the complex variable $z_1 = x + iy$, $z_2 = x - iy$ in a plane, we have several remarkable systems comprised in the form

$$\phi(z_1)dz_1 + \phi(z_2)dz_2 = 0.$$

Thus

$$\frac{dz_1}{z_1} \pm \frac{dz_2}{z_2} = 0$$

represents concentric circles and their radii;

$$\frac{dz_1}{\sqrt{(c^2 - z_1^2)}} \pm \frac{dz_2}{\sqrt{(c^2 - z_2^2)}} = 0$$

represents confocal conics, the property that the tangents bisect the angle between the focal radii immediately following from this equation;

$$\frac{dz_1}{\sqrt{Z_1}} \pm \frac{dz_2}{\sqrt{Z_2}} = 0$$

represents confocal Cartesian ovals, where

$$Z = (z - a)(z - b)(z - c);$$

and

$$\frac{mdz_1}{\sqrt{Z_1}} \pm \frac{ndz_2}{\sqrt{Z_2}} = 0$$

represents confocal bicircular quartics, if Z_1 and Z_2 are biquadratic functions of z_1, z_2 respectively, whose roots are homographically connected, and m, n are special functions of the roots. These results were, I believe, first noticed by Darboux.

Further, I may notice that

$$\frac{dz_1}{(Z_1)^{\frac{2}{3}}} + \frac{dz_2}{(Z_2)^{\frac{2}{3}}} = 0,$$

where $Z = (z - a)(z - b)(z - c)$ represents a system of curves of the sixth order, such that the three of the system passing through a given point cut at angles of 120°. I have investigated several properties of this system in an article contributed to vol. 15 of the *Messenger of Mathematics*.

Other remarkable differential equations are those of systems of lines in elliptic co-ordinates; namely, if p, q are the roots of

$$\frac{x^2}{a+p}+\frac{y^2}{b+p}=1,$$

then

$$\frac{dp}{\sqrt{\{(a+p)(b+p)(\alpha+p)\}}}\pm\frac{dq}{\sqrt{\{(a+q)(b+q)(\alpha+q)\}}}=0$$

represents all the lines touching the conic $p+\alpha=0$; and further, if p, q, r are the roots of

$$\frac{x^2}{a+p}+\frac{y^2}{b+p}+\frac{z^2}{c+p}=1,$$

then

$$\Sigma\frac{dp}{\sqrt{\{(a+p)(b+p)(c+p)(\alpha+p)(\alpha'+p)\}}}=0,$$

$$\Sigma\frac{pdp}{\sqrt{\{(a+p)(b+p)(c+p)(\alpha+p)(\alpha'+p)\}}}=0,$$

represent all the lines touching the two confocals $p+\alpha=0$, $p+\alpha'=0$.

The latter equations involve the first class of hyperelliptic integrals, and thus demonstrate the fact of equivalent algebraic relations. These equations, originally due to Liouville, were seen by Haedenkamp to hold for n dimensions. The latter thus deduced the algebraic integrals of the whole class of hyperelliptic integrals.

Another system of differential equations may be found by using the parameters of the two tangents drawn from a point to a conic as co-ordinates. Thus if μ_1, μ_2 are the roots of $\mu^2x-2\mu z+y=0$, a tangent of $z^2-xy=0$,

$$\frac{d\mu_1}{\sqrt{M_1}}\pm\frac{d\mu_2}{\sqrt{M_2}}=0,$$

where $M=(\mu-a)(\mu-b)(\mu-c)(\mu-d)$, represents the system of conics touching the four tangents of $z^2-xy=0$, whose parameters are a, b, c, d. Darboux has made an extensive use of this kind of co-ordinates.

Now I propose to consider here a few forms of differential equations, similar to those just given, which represent systems of conics both in a plane and in space.

First of all, in a plane, I consider the three systems of conics having double contact with two fixed conics. Let the latter be referred to their common self-conjugate triangle, and let us consider the system of conics (touching four fixed lines)

$$U_p = \frac{x^2}{a+p} + \frac{y^2}{b+p} + \frac{z^2}{c+p} = 0; \tag{1}$$

then if p, q are the roots of this equation, the differential equation

$$\frac{dp}{\sqrt{[(a+p)(b+p)(\alpha+p)(\alpha'+p)]}} \pm \frac{dq}{\sqrt{[(a+q)(b+q)(\alpha+q)(\alpha'+q)]}} = 0 \tag{2}$$

represents a system of conics having double contact with the conics obtained by putting $\alpha+p=0$ and $\alpha'+p=0$ in (1). For, by the theory of elliptic functions, the integral of the equation can be written as a linear relation between any three of the quantities $\sqrt{\{(a+p)(a+q)\}}$, $\sqrt{\{(b+p)(b+q)\}}$, $\sqrt{\{(\alpha+p)(\alpha+q)\}}$, $\sqrt{\{(\alpha'+p)(\alpha'+q)\}}$; but the first two of these quantities are proportional to x, y, respectively, and the last two to $\sqrt{U_\alpha}$ and $\sqrt{U_\alpha'}$. Thus we get the equivalent equations

$$lx + my + n\sqrt{U_\alpha} = 0,$$
$$l'x + m'y + n'\sqrt{U_\alpha'} = 0,$$

which represent a conic having double contact with U_α and U_α', respectively.

Again, by using the factors $(b+p)(c+p)$ and $(a+p)(c+p)$ successively, instead of $(a+p)(b+p)$ in (2), we get the differential equations of the two other systems of conics.

As a particular case of this result, suppose we consider the ordinary plane elliptic co-ordinates, viz., those determined by

$$\frac{x^2}{\mu^2} + \frac{y^2}{\mu^2 - c^2} = 1, \qquad \frac{x^2}{\nu^2} - \frac{y^2}{c^2 - \nu^2} = 1;$$

then

$$\left.\begin{aligned}
\frac{d\mu}{\sqrt{[(\mu^2-a^2)(\mu^2-a'^2)]}} \pm \frac{d\nu}{\sqrt{[(a^2-\nu^2)(a'^2-\nu^2)]}} &= 0,\\
\frac{d\mu}{\sqrt{[(\mu^2-a^2)(\mu^2-a'^2)(\mu^2-c^2)]}} \pm \frac{d\nu}{\sqrt{[(a^2-\nu^2)(\nu^2-a'^2)(c^2-\nu^2)]}} &= 0,\\
\frac{\mu\,d\mu}{\sqrt{[(\mu^2-a^2)(\mu^2-a'^2)(\mu^2-c^2)]}} \pm \frac{\nu\,d\nu}{\sqrt{[(a^2-\nu^2)(\nu^2-a'^2)(c^2-\nu^2)]}} &= 0,
\end{aligned}\right\} \tag{3}$$

represent the three systems of conics having double contact with two confocal conics.

If we make $a' = \infty$, which corresponds to the two imaginary points at infinity, we get

$$\left.\begin{aligned} \frac{d\mu}{\sqrt{(a^2-\mu^2)}} \pm \frac{d\nu}{\sqrt{(a^2-\nu^2)}} = 0, \\ \frac{\mu d\mu}{\sqrt{\{(\mu^2-a^2)(\mu^2-c^2)\}}} \pm \frac{\nu d\nu}{\sqrt{\{(a^2-\nu^2)(c^2-\nu^2)\}}} = 0, \end{aligned}\right\} \quad (4)$$

as the differential equations of the two systems of circles having double contact with the conic $\mu = a$. The third equation gives in the same case the system of tangents to $\mu = a$.

Proceeding now to the three dimensions, let us consider the system of quadrics (inscribed in the same developable)

$$U_p = \frac{x^2}{a+p} + \frac{y^2}{b+p} + \frac{z^2}{c+p} + \frac{u}{d+p} = 0; \qquad (5)$$

then, if p, q, r are the roots of this equation,

$$\left.\begin{aligned} \Sigma \frac{dp}{\sqrt{\{(a+p)(b+p)(c+p)(\alpha+p)(\alpha'+p)(\alpha''+p)\}}} = 0, \\ \Sigma \frac{pdp}{\sqrt{\{(a+p)(b+p)(c+p)(\alpha+p)(\alpha'+p)(\alpha''+p)\}}} = 0, \end{aligned}\right\} \quad (6)$$

represents a conic having double contact with the three quadrics U_α, U_α', U_α'', for by the theory of the first class of hyperelliptic integrals the integrals can be expressed as linear relations between any three of the six quantities

$$\begin{aligned} &\sqrt{\{(a+p)(a+q)(a+r)\}}, \qquad &&\sqrt{\{(\alpha-p)(\alpha-q)(\alpha-r)\}}, \\ &\sqrt{\{(c+p)(c+q)(c+r)\}}, &&\sqrt{\{(\alpha'-p)(\alpha'-q)(\alpha'-r)\}}, \\ &\sqrt{\{(b+p)(b+q)(b+r)\}}, &&\sqrt{\{(\alpha''-p)(\alpha''-q)(\alpha''-r)\}}. \end{aligned}$$

Now the first three of these are proportional to x, y, z, respectively, and the three last are proportional to $\sqrt{U_\alpha}$, $\sqrt{U_\alpha'}$, $\sqrt{U_\alpha''}$. Thus we get

$$\begin{aligned} lx + my + nz &= 0, \\ l'x + m'y + n'\sqrt{U_\alpha} &= 0, \\ l''x + m''y + n''\sqrt{U_\alpha'} &= 0, \\ l'''x + m'''y + n'''\sqrt{U_\alpha''} &= 0, \end{aligned}$$

and these equations obviously represent a conic having double contact with the quadrics U_α, U_α', U_α''.

Again, by interchanging the constants a, b, c, d we obtain three other systems similar to (6).

Thus we have demonstrated that there are four different systems of conics that can be described to have double contact with three quadrics inscribed in the same developable.

As confocal quadrics are of this kind, their equations being of the form

$$\frac{x^2}{a+p}+\frac{y^2}{b+p}+\frac{z^2}{c+p}=1,$$

we obtain the differential equations of conics described to have double contact with three confocal quadrics by making $p=\infty$ in the three equations similar to (6), each of which involves d.

As a special case, suppose $\alpha''=\infty$, which corresponds to the imaginary circle at infinity: then the conics become circles having double contact with two confocal quadrics.

Now equations (6) in this case become simply the differential equations of lines touching the two confocals, but the three similar equations give those of the derived systems, viz.:

$$\Sigma\frac{dp}{\sqrt{\{(a+p)(b+p)(\alpha+p)(\alpha'+p)\}}}=0,$$
$$\Sigma\frac{pdp}{\sqrt{\{(a+p)(b+p)(\alpha+p)(\alpha'+p)\}}}=0,$$

and the two other similar equations obtained by interchanging a, b, c. Thus there are three systems of circles which can be described to have double contact with two confocal quadrics. Furthermore, either of these quadrics can degenerate to a focal conic. Thus, for instance, since $p+a=0$, $p+b=0$ give the two focal conics, we have

$$\Sigma\frac{\pm\,dp}{(a+p)(b+p)}=0,\qquad \Sigma\frac{\pm\,pdp}{(a+p)(b+p)}=0;$$
$$\Sigma\frac{dp}{(a+p)\sqrt{\{(b+p)(c+p)\}}}=0,\qquad \Sigma\frac{pdp}{(a+p)\sqrt{\{(b+p)(c+p)\}}}=0;$$
$$\Sigma\frac{dp}{(b+p)\sqrt{\{(a+p)(c+p)\}}}=0,\qquad \Sigma\frac{pdp}{(b+p)\sqrt{\{(a+p)(c+p)\}}}=0;$$

as the differential equations of circles described to intersect twice each of the two focal conics.

In each of the preceding equations we have $dp:dq:dr=A:\pm B:\pm C$, say; that is, there are, in consequence of the double signs, four tangents to the curves of the system that pass through a given point. Thus there are four curves of the system that pass through a point, and from the differential equations we can find properties of these curves, and loci connected with their intersection. Thus in elliptic co-ordinates, the normals to the confocal quadrics being at right angles to each other, it follows that the tangents to the four conics of the aforementioned systems that pass through a point are equally inclined to the normal of one of the quadrics; that is, if α, β, γ are the angles which one of the tangents makes with the normals of the confocal quadrics, we shall have $\cos\alpha$, $\cos\beta$, $\cos\gamma$ respectively proportional to $\lambda, \pm\mu, \pm\nu$, say, for the tangents of the four conics of the system. Hence, if two of the conics should cut at right angles, we should have $\lambda^2 + \mu^2 = \nu^2$, from which we can derive the equation of the locus in elliptic co-ordinates.

I now proceed to consider the problem: to find the differential equations of the system of conics touching six fixed planes in space.

Now there is a definite unique twisted curve of the third degree which may be described to osculate the six fixed planes. Hence, since an osculating plane of the twisted cubic can be written in the form

$$X_\mu = \mu^3 x + \mu^2 y + \mu z + u = 0,$$

it follows that the six planes can be obtained by substituting six quantities $a_1, a_2, a_3, a_4, a_5, a_6$ in this equation in place of μ. Now let us consider the differential equations

$$\Sigma\frac{d\mu}{\sqrt{M}} = 0, \qquad \Sigma\frac{\mu d\mu}{\sqrt{M}} = 0, \tag{7}$$

where the summation refers to the parameters μ_1, μ_2, μ_3 of the three osculating planes which can be drawn through a point, and

$$M = (\mu - a_1)(\mu - a_2)(\mu - a_3)(\mu - a_4)(\mu - a_5)(\mu - a_6).$$

Then, by the theory of the first class of hyperelliptic integrals, we have as the integrals a series of linear relations connecting the six quantities

$$\sqrt{\{(a_1-\mu_1)(a_1-\mu_2)(a_1-\mu_3)\}}, \quad \sqrt{\{(a_2-\mu_1)(a_2-\mu_2)(a_2-\mu_3)\}}, \\ \dots, \quad \sqrt{\{(a_6-\mu_1)(a_6-\mu_2)(a_6-\mu_3)\}}.$$

But, from the equation of the osculating plane, the expression $(a_1 - \mu_1)(a_1 - \mu_2)(a_1 - \mu_3)$ is proportional to the osculating plane corresponding to the parameter a_1, and similarly for the other expressions. Thus we have linear relations connecting every three of the radicals $\sqrt{X_{a_1}}, \sqrt{X_{a_2}}, \dots, \sqrt{X_{a_6}}$;

or we may take each of these proportional to $p_1 + q_1t$, $p_2 + q_2t$, ..., $p_6 + q_6t$, where t is an arbitrary parameter. Hence we get

$$\lambda X_{a_1}=(p_1+q_1t)^2,\quad \lambda X_{a_2}=(p_2+q_2t)^2,\quad \ldots,\quad \lambda X_{a_6}=(p_6+q_6t)^2.$$

Now these equations obviously represent a curve of the second degree, viz., a conic section, touching the six planes X_{a_1}, X_{a_2}, ..., X_{a_6}. Consequently the differential equation (7) represents the system of conics described to touch the six planes. Since there are four values of the ratios $d\mu_1 : d\mu_2 : d\mu_3$, it follows that there are four conics of the system passing through an arbitrary point of space.

In a paper contributed to *The Proceedings of the London Mathematical Society*, vol. 18, p. 202, I showed that if certain conditions were satisfied a doubly infinite number of polygons could be circumscribed about two confocal quadrics so as to have their vertices on another confocal quadric. This result was obtained by means of the theory of the first class of hyperelliptic integrals. Similar reasoning could be employed in the case of the conics that I have been considering. Thus, under certain conditions, we could have an infinite number of curvilinear polygons formed by conics having double contact with two confocal quadrics and intersecting on another confocal quadric. There would also be infinite systems of curvilinear polygons in the other cases that I have investigated.

An interesting question arises in connection with the preceding results, viz., as to the reality of the curves of the different systems. Thus, for instance, in the case of two confocal ellipses represented in elliptic co-ordinates by $\mu = a$, $\mu = a'$, the conics belonging to the system

$$\frac{d\mu}{\sqrt{\{(\mu^2-a^2)(\mu^2-a'^2)\}}} \pm \frac{d\nu}{\sqrt{\{(a^2-\nu^2)(a'^2-\nu^2)\}}} = 0$$

are not real, because μ must obviously lie between a and a', and thus $(\mu^2-a^2)(\mu^2-a'^2)$ is negative, while $(a^2-\nu^2)(a'^2-\nu^2)$ is positive.

But the two other systems contain real conics, and in other cases there seem to be only two systems with real conics.

A case, however, not already noticed, in which the three systems are real, deserves to be mentioned. One of the fixed conics is, however, imaginary, namely, the imaginary circle at infinity on a sphere. Consider the confocal cones

$$\frac{x^2}{\mu^2}+\frac{y^2}{\mu^2-b^2}=\frac{z^2}{c^2-\mu^2},$$
$$\frac{x^2}{\nu^2}-\frac{y^2}{b^2-\nu^2}=\frac{z^2}{c^2-\nu^2},$$

intersecting the sphere $x^2 + y^2 + z^2 = 1$ in confocal spheroconics; then the differential equations

$$\frac{d\mu}{\sqrt{\{(\mu^2 - b^2)(\mu^2 - a^2)\}}} \pm \frac{d\nu}{\sqrt{\{(b^2 - \nu^2)(a^2 - \nu^2)\}}} = 0,$$

$$\frac{d\mu}{\sqrt{\{(c^2 - \mu^2)(a^2 - \mu^2)\}}} \pm \frac{d\nu}{\sqrt{\{(c^2 - \nu^2)(a^2 - \nu^2)\}}} = 0,$$

$$\frac{\mu d\mu}{\sqrt{\{(\mu^2-b^2)(c^2-\mu^2)(\mu^2-a^2)\}}} \pm \frac{\nu d\nu}{\sqrt{\{(b^2-\nu^2)(c^2-\nu^2)(a^2-\nu^2)\}}} = 0,$$

represent the three systems of circles having double contact with the sphero-conic $\mu = a$, and these it is easy to see from geometrical considerations are real. Circles of course may be considered as conics having double contact with $\mu = \infty$, or $x^2 + y^2 + z^2 = 0$. By projection then on a plane it would appear that we could have three real systems of conics having double contact with one real conic and one imaginary conic.

It would be interesting also to consider the reality of the conics having double contact with three quadrics. For instance, it does not seem possible to have any real conics having real double contact with three confocal ellipsoids. But I must defer the consideration of this question to another opportunity.

ASYMPTOTIC LINES ON A CIRCULAR RING.*

BY PROFESSOR H. MASCHKE.

THE equation of the asymptotic lines of a tore as well as the expression of the length of the arc of these curves leads to elliptic functions. The formulas appear in a remarkably simple form.†

Assuming the radius CA of the rotating circle $= 1$, and denoting the distance CO of its centre from the axis of rotation by λ, the tore is represented by these equations:

$$x = v \cos u, \quad y = v \sin u, \quad z = \sqrt{1 - (v - \lambda)^2},$$

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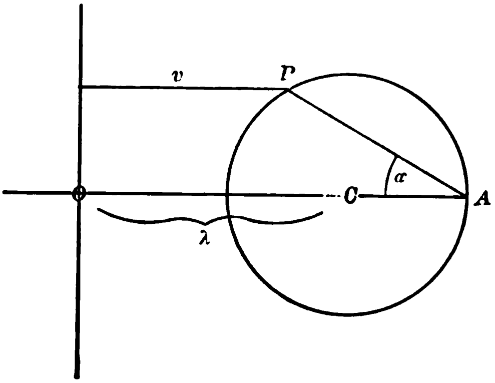
* Read before the AMERICAN MATHEMATICAL SOCIETY at its Second Summer Meeting, Springfield, Mass., August 28, 1895.

† I have not found any literature with regard to this subject excepting a short notice in the *Fortschritte der Mathematik*, 1889, page 765, where reference is made to T. MOTODA, "Asymptotic lines on a circular ring," *Tokio Math. Soc.*, vol. 4, 217–219, a paper which has been inaccessible to me.

where u is the angle between the plane of the revolving circle and the XZ-plane and v the distance of a point P of the surface from the Z-axis, the axis of rotation.



Then the differential equation of the asymptotic curves is this:

$$\left(\frac{dv}{du}\right)^2 = v(\lambda - v)(1 + \lambda - v)(1 - \lambda + v).$$

From this equation we find v in terms of Weierstrass's $\wp$-function

$$v = \frac{2\lambda(6\wp u - 1)}{12\wp u + 1},$$

the invariants of $\wp u$ being

$$g_2 = \frac{1 + 3\lambda^2}{12}, \quad g_3 = \frac{1 - 9\lambda^2}{12.18},$$

and

$$e_1 = \frac{3\lambda - 1}{12}, \quad e_2 = \frac{1}{6}, \quad e_3 = -\frac{3\lambda + 1}{12}.$$

In terms of σ-functions there is:

$$v = \frac{2\lambda\sigma_2{}^2 u}{\sigma_1{}^2 u + \sigma_2{}^2 u},$$

and denoting $\sphericalangle OAP = \alpha$,

$$\tan \alpha = \frac{\sigma_1 u}{\sigma_2 u}.$$

The length of the arc of an asymptotic curve is given by the integral

$$s = \sqrt{\lambda} \int_v^{\lambda} \frac{dv}{\sqrt{(\lambda - v)(1 + \lambda - v)(1 - \lambda + v)}}.$$

Introducing the ℘-function with the invariants $g_2 = \frac{1}{4\lambda^2}$, $g_3 = 0$; and $e_1 = \frac{1}{4\lambda}$, $e_2 = 0$, $e_3 = -\frac{1}{4\lambda}$, $\left(k^2 = \frac{e_2 - e_3}{e_1 - e_3} = \frac{1}{2}\right)$, we obtain:

$$\lambda - v = \frac{1}{4\lambda} \cdot \frac{1}{\wp s}.$$

UNIVERSITY OF CHICAGO, *August* 23, 1895.

ON A GENERALIZATION OF WEIERSTRASS'S EQUATION WITH THREE TERMS.

BY PROFESSOR F. MORLEY.

THE expression

$$\prod_{\lambda=1}^{n} \sigma(u - b_\lambda)/\sigma(u - a_\lambda)$$

is an elliptic function of u if

$$\Sigma a_\lambda = \Sigma b_\lambda.$$

The sum of the residues is zero; that is,

$$(1) \qquad \sum_{\lambda=1}^{n} \frac{\sigma(a_\lambda - b_1) \ldots \sigma(a_\lambda - b_\lambda) \ldots \sigma(a_\lambda - b_n)}{\sigma(a_\lambda - a_1) \ldots \quad 1 \quad \ldots \sigma(a_\lambda - a_n)} = 0.$$

Being now only concerned with differences, we can, by a suitable addition to each a and b, write

$$(2) \qquad \Sigma a_\lambda = \Sigma b_\lambda = 0.$$

When $n = 2$, the equation (1) is in no way characteristic of the σ-function, but is true of any odd function.

When $n = 3$, (1) becomes

$$(3) \qquad \begin{aligned} &\sigma(a_1 - b_1)\sigma(a_1 - b_2)\sigma(a_1 - b_3)\sigma(a_2 - a_3) \\ + &\sigma(a_2 - b_1)\sigma(a_2 - b_2)\sigma(a_2 - b_3)\sigma(a_3 - a_1) \\ + &\sigma(a_3 - b_1)\sigma(a_3 - b_2)\sigma(a_3 - b_3)\sigma(a_1 - a_2) = 0. \end{aligned}$$

To identify this with the equation of three terms as given by Weierstrass, write

$$\begin{aligned} a &= b_1 - a_1, \\ b &= a_1 - b_2, \\ c &= a_1 - b_3, \\ d &= a_2 - a_3, \end{aligned}$$

and use the notation given by Harkness and Morley, Theory of Functions, p. 313.

We have

$$\begin{aligned} a' &= b_1 - a_3, & a'' &= b_1 - a_2, \\ b' &= a_3 - b_2, & b'' &= a_2 - b_2, \\ c' &= a_3 - b_3, & c'' &= a_2 - b_3, \\ d' &= a_1 - a_2, & d'' &= a_3 - a_1, \end{aligned}$$

so that our equation (3) may be written

$$\sigma a \sigma b \sigma c \sigma d + \sigma a' \sigma b' \sigma c' \sigma d' + \sigma a'' \sigma b'' \sigma c'' \sigma d'' = 0,$$

which is equation (48) of the page referred to.

For higher values of n the equation (1) appears to be the simplest of the various possible extensions of the equation with three terms.

NOTES.

AMONG recent academic appointments are the following: Professor Samuel M. Barton, recently of the Virginia Agricultural and Mechanical College, has been appointed professor of mathematics at the University of the South, Sewanee, Tenn.; Dr. Alexander Macfarlane, formerly professor of physics at the University of Texas, has accepted a lectureship in electrical engineering at Lehigh University, South Bethlehem, Pa.; Dr. E. B. Van Vleck has resigned an instructorship at the University of Wisconsin in order to become associate professor of mathematics at Wesleyan University, Middletown, Conn.; Professor C. A. Waldo has been called from De Pauw University to the chair of mathematics at Purdue University, La Fayette, Ind.; Professor W. J. Hussey, of Stanford University, has been appointed astronomer at the Lick Observatory, to succeed Professor E. E. Barnard, who has been called to a chair of astronomy at the University of Chicago.

DR. PAUL STÄCKEL, of Halle, has been appointed professor extraordinarius at Königsberg, and Professor V. Eberhardt, of Königsberg, has been called to Halle.

WE are grieved to learn of the death of Dr. Ernst Ritter, on August 23, of typhoid fever, at the government hospital on Ellis Island in New York harbor. He was on his way from Germany to begin his duties as assistant professor of mathematics at Cornell University.

THE STEINER PRIZE.—The Academy of Sciences of Berlin at the Leibnitz Meeting, July 3, 1890, announced the following problem for the Steiner Prize:—The solution of an important problem in the theory of lines of curvature of surfaces; preferably, the determination of the conditions under which the lines of curvature of algebraic surfaces are algebraic curves. No paper on this subject was received. In accordance with the terms of the Steiner foundation, the Academy has utilized the prize, thus unawarded, for the purpose of recognizing certain important geometrical contributious published during the past few years. One half of it has been awarded to Dr. Sigmund Gundelfinger, professor at the polytechnic school at Darmstadt, for his remarkable work leading to the fundamental investigation and development of the methods introduced into geometry by Hesse; the other half to Dr. Friedrich Schottky, professor at the University of Marburg, who has afforded valuable assistance in a series of most important special problems of geometry by demonstrating their relations to the theory of abelian functions of two, three, and four variables.

For the year 1900 the Academy announces the following problem for the Steiner Prize:—To completely solve any important, hitherto unsolved problem relating to the theory of curved surfaces, taking into account so far as possible the methods and principles evolved by Steiner. It is required that sufficient analytical explanations shall accompany the geometrical investigations to verify the correctness and completeness of the solution. Without wishing to limit the choice of subject the Academy takes the opportunity to call attention to the special problems to which Steiner has referred in his general remark at the end of his second paper on maximum and minimum in figures in a plane, on a sphere, and in space. For the solution of the problem a prize of 4000 marks is offered, with an additional sum of 2000 marks. Papers offered in competition may be written in German, French, English, Italian, or Latin, and must be submitted before December 31, 1899. The result will be announced at the Leibnitz Meeting of 1900.

THE attendance at the meeting of the British Association for the Advancement of Science, held at Ipswich, September 11–18, was somewhat smaller than at the Oxford meeting last year. Professor W. M. Hicks was president of Section A, representing mathematics and physics. His address, the subject of which was "The fluid theories of ether and matter," appears in *Nature* of September 12.

THE French Association for the Advancement of Science met at Bordeaux, August 4, under the presidency of Émile Trélat. There were more than 800 members present. According to *La Nature* the following were among the most interesting mathematical papers read:—Mr. TORRIS, *On an algebraic machine;* Mr. COCCOZ, *On magic squares;* Mr. GUIMARET, *On vortices;* Mr. COLLIGNON, *On certain questions in geometry.*

WE take pleasure in announcing the following work as in press: *Theory of functions of a complex variable, with special reference to Riemann surfaces.* By Dr. H. DURÈGE, late Professor in the University of Prague. Authorized translation from the fourth German edition. By GEORGE EGBERT FISHER, M.A., Ph.D., and ISAAC J. SCHWATT, Ph.D., of the University of Pennsylvania, Philadelphia. The work is being published by the translators.

AMONG the courses for the winter semester at Berlin are the following:—Professor Fuchs: Analytical mechanics; Abelian functions;—Professor Frobenius: Theory of numbers;—Professor Schwarz: Synthetic geometry; Selected chapters from the theory of analytical functions; Theory of elliptic functions, making use of his edition of "Formeln und Lehrsätze zum Gebrauche der elliptischen Functionen";—Professor Knoblauch: Analytical geometry; Definite integrals; Mathematical exercises for younger students;—Professor Hensel: Algebra; Integral calculus; Exercises in algebra;—Professor Hettner: Theory of probabilities;—Dr. Schlesinger: Theory of Fourier's series and spherical harmonics, with applications to questions of mathematical physics; Theory of substitution groups, with applications to linear differential equations;—Dr. Kötter: Geometry of circle and sphere systems, with special reference to bicircular curves and surfaces of the fourth order; Differential calculus and introduction to analysis; Exercises;—Dr. Hoppe: Analytical geometry, in connection with his text-book; Differential calculus and series, in connection with his text-book;—Dr. Wien: Hydrodynamics.

AMONG the courses for the winter semester at Leipsic are the following: Professor Scheibner: On the hypergeometric series and Eulerian integrals;—Professor Neumann: Selected chapters from mathematical physics; Discussion of mathematical problems;—Professor Lie: Introduction to metrical geometry; On differential equations which admit infinitesimal transformations; Exercises in mathematical seminarium;—

Professor Mayer: General introduction to the differential and integral calculus; Exercises;—Professor Engel: Introduction to analytical mechanics; Higher algebra; Conference on mathematical problems;—Dr. Scheffers: Introduction to the theory of linear differential equations; Descriptive geometry accompanied by exercises; Kinematics (theory of mechanisms).

NEW PUBLICATIONS.

I. HIGHER MATHEMATICS.

Ahrens (W.). Ueber eine Gattung von n-fach periodischen Functionen von n reellen Veränderlichen. [Diss.] Rostock, 1895. 8vo. 36 pp. Mk. 1.20

Arzelà (C.). Sull' integrabilità delle equazioni differenziali ordinarie; nota letta alla R. Accademia delle scienze dell' Istituto di Bologna nella sessione del 24 Marzo 1895. Bologna, Gamberini e Parmeggiani, 1895. 4to. 16 pp.

Bäcklin (G.). Allmän Mathematik eller s. k. Proportionslära. Upsala, 1893. 4to. 22 pp. Illustrated.

Ballauf. Einige Hauptsätze aus der Lehre von den Kegelschnitten in elementarer Behandlung. Theil I. Gebweiler, 1894. 4to. 13 pp. 1 plate. Mk. 1.50.

Biermann (O.). Elemente der höheren Mathematik. Vorlesungen zur Vorbereitung des Studiums der Differentialrechnung, Algebra, und Functionentheorie. Leipzig, Teubner, 1895. 8vo. 12 and 381 pp. Mk. 10.00.

Bockwoldt (G.). Die analytische Geometrie in der Prima des Gymnasiums. Theil II. Neustadt, 1895. 8vo. 16 pp. 2 plates. Mk. 1.20

Bode (J.). Die Grundlagen der höheren Mathematik in schulmässiger Behandlung. Mülheim, 1895. 8vo. 37 pp. Mk. 1.20

Breuer (P. J.). Die gemeinen Logarithmen. Wipperfürth, 1894. 4to. 19 pp.

Cayley (A.). An elementary treatise on elliptic functions. 2d edition. London, Bell, 1895. 8vo. 390 pp. 15s.

Crivetz (T.). Essay on the postulate of Euclid. (*In Rumanian.*) Bucharest, Socecü, 1895. 8vo. 40 pp. Illustrated. Mk. 2.00

Engel (F.). See **Stäckel** (P.).

Epstein (S. S.). Die vier Rechnungsoperationen mit Bessel'schen Functionen, nebst einer geschichtlichen Einleitung. [Diss.] Bern, 1894. 8vo. 54 pp. Mk. 1.50

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OF THE

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Continuation of the Bulletin of the New York Mathematical Society.

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EDITED BY
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2nd Series, Vol. II. No. 2.

November, 1895.

NEW YORK:
Published for the Society
By MACMILLAN & CO.
1895.

Entered at the Post-office at New York, N. Y., as second-class matter.
Published Monthly, except August and September.
Whole No. 42. $5.00 a year.

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BULLETIN OF THE AMERICAN MATHEMATICAL SOCIETY,

41 East 49th Street, New York City.

The following dates have been fixed for the meetings of the Society:

				P.M.					P.M.
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"	Nov.	30,	"	"	"	Mar.	28,	"	"
Fri.,	Dec.	27,	"	"	"	April	25,	"	"
Sat.,	Jan.	25,	1896,	"	"	May	23,	"	"

The Meeting of Friday, December 27, 1895, *will be the Annual Meeting of the Society for the election of officers.*

PRINTED BY ROBERT DRUMMOND, NEW YORK.

BULLETIN

OF THE

AMERICAN

MATHEMATICAL SOCIETY.

CONTINUATION OF THE BULLETIN OF THE NEW YORK MATHEMATICAL SOCIETY.

A HISTORICAL AND CRITICAL REVIEW
OF MATHEMATICAL SCIENCE.

EDITED BY
THOMAS S. FISKE, ALEXANDER ZIWET, FRANK MORLEY.

2ND SERIES, VOL. II. NO. 2.

NOVEMBER, 1895.

NEW YORK:
PUBLISHED FOR THE SOCIETY
BY MACMILLAN & CO.
1895.

Entered at the Post-office at New York, N. Y., as second-class matter.
PUBLISHED MONTHLY, EXCEPT AUGUST AND SEPTEMBER.

Whole No. 42.

$5.00 a Year.

CONCERNING JORDAN'S LINEAR GROUPS.

Presented to the American Mathematical Society, August 28, 1895.

BY ELIAKIM HASTINGS MOORE.

Introduction.

I PRESENT to the AMERICAN MATHEMATICAL SOCIETY to-day a continuation of the paper* presented last November, entitled *The group of holoedric transformation into itself of a given group.* To recall briefly: The given (abstract) group G_n of order n has the elements $s_1 =$ identity, $s_2, \ldots s_n$. The substitution-group Γ^n of transformation of G_n into itself is *the* substitution-group on the n letters $s_1, \ldots s_n$ which leaves invariant the multiplication-table for G_n. Letters s which are conjugate with one another under Γ^n must as elements of G_n have the same period. Thus, $s_1 =$ identity is invariant, and Γ^n is really Γ^{n-1} on the $n-1$ letters $s_2, \ldots s_n$.

We are to consider to-day the case that Γ^{n-1} *is transitive on the* $n-1$ *letters* $s_2, \ldots s_n$. Then the $n-1$ elements $s_2, \ldots s_n$ of G_n have the *same* period, which must then be a *prime* p. Hence G_n has the order $n = p^{n'}$. Every group $G_{n=p^{n'}}$ has, in accordance with an important (Sylow's) theorem,† at least one element different from identity commutative with every element of the group. This property of the element may be read out of the multiplication-table for $G_{n=p^{n'}}$, and is hence invariant under Γ^{n-1}. But Γ^{n-1} is transitive on the $n-1$ letters $s_2, \ldots s_n$. Hence *every element of* $G_{n=p^{n'}}$ *is commutative with every other element. Our given group* G_n *is then the Abelian* $G_{p^{n'}}$, or rather, omitting the $'$, G_{p^n} *with* n *generating elements, each of order* p, *and commutative with one another.* It will cause no confusion if we refer to it hereafter simply as *the Abelian* G_{p^n}.

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* *Bulletin of the American Mathematical Society*, ser. 2, vol. 1, pp. 61–66, Dec. 1894.

Mr. HÖLDER explained this notion of the group of holoedric transformations into itself of a given group, for use in his memoir: *Die Gruppen der Ordnungen* p^3, pq^2, pqr, p^4 (*Mathematische Annalen*, vol. 43, pp. 301–412; see pp. 313, 314), which bears the date March 28, 1893. We, however, hit on the notion independently of each other; see the foot-note (**) of p. 66 of my former paper.

† SYLOW: *Mathematische Annalen*, vol. 5, p. 588.

§ 1.

The group $\Gamma^{p^n}_{\Omega(p^n)}$ *of holoedric transformation into itself of the Abelian group* G_{p^n} *is Jordan's linear homogeneous substitution-group of degree* p^n, $LHG^{p^n}_{\Omega(p^n)}$.

For the Abelian G_{p^n} we take the n generators

$$(1) \qquad a_i \qquad (i=1, 2, \ldots n)$$

with the complete system of generating relations.

$$(2) \qquad a_i^p = 1, \qquad a_i a_j = a_j a_i \qquad (i, j = 1, 2, \ldots n)$$

and have as the general element

$$(3) \qquad s_K = s_{k_1, k_2, \ldots k_n} = a_1^{k_1} a_2^{k_2} \ldots a_n^{k_n} = \prod_i a_i^{k_i} \qquad (i = 1, 2, \ldots n)$$

where the suffixes and exponents k are integers taken modulo p, and where K is a symbol standing for $(k_1, k_2, \ldots k_n)$.

The general multiplication equation is

$$(4) \qquad s_{K_1} s_{K_2} = s_{K_3}, \quad s_{k_{11}, k_{12}, \ldots k_{1n}} \cdot s_{k_{21}, k_{22}, \ldots k_{2n}} = s_{k_{31}, k_{32}, \ldots k_{3n}},$$

where

$$(5) \qquad K_1 + K_2 = K_3, \quad k_{1i} + k_{2i} = k_{3i}. \qquad (i = 1, 2 \ldots n)$$

It turns out that the general substitution σ_G of $\Gamma^{p^n}_{\Omega(p^n)}$ rereplaces $s_X = s_{x_1, x_2, \ldots x_n}$ by $s_{X'} = s_{x'_1, x'_2, \ldots x'_n}$, where

$$(6) \quad X' = GX, \qquad x'_i = \sum_{j=1}^{j=n} g_{ij} x_j \qquad (|g_{ij}| \neq 0) \qquad (i, j = 1, 2, \ldots n),$$

where G is a symbol for the matrix

$$(7) \qquad G = (g_{ij}) \qquad (i, j = 1, 2, \ldots n)$$

whose elements g_{ij} are integers taken modulo p. [To follow the customary notation we should write *congruences* (*modulo* p) everywhere instead of *equations*. But in group-theoretic applications such as the present, it is much better to breathe the *spirit* of the congruence once for all into the *definitions* of the symbols and operations.] Hence, indeed, $\Gamma^{p^n}_{\Omega(p^n)}$ is *Jordan's* linear homogeneous substitution-group* of degree p^n, $LHG^{p^n}_{\Omega(p^n)}$, of order †

$$(8) \qquad \Omega(p^n) = (q^n - 1)(q^n - q)(q^n - q^2) \cdots (q^n - q^{n-1}).$$

---

* Jordan: *Traité des substitutions*, p. 92, 1870.

† Jordan: loc. cit., p. 97.

This identification of the $\Gamma^{p^n}_{\Omega(p^n)}$ of the Abelian G_{p^n} with the $LHG^{p^n}_{\Omega(p^n)}$ I obtain first by holding the G_{p^n} as an *abstract* group; I omit the details of this identification. We may however take the G_{p^n} *concretely* as the *regular* Abelian substitution-group $G^{p^n}_{p^n}$ on the p^n letters $s_X = s_{x_1, x_2, \ldots x_n}$; the general element (3) $s_K = s_{k_1, k_2, \ldots k_n}$ then (4, 5) replaces s_X by $s_{X'}$, where

$$(9) \qquad X' = X + K, \quad x'_i = x_i + k_i \qquad (i = 1, 2, \ldots n).$$

We thus win direct contact with Mr. *Jordan's* work. The $G^{p^n}_{p^n}$ (9) is within the symmetric substitution-group on the p^n letters s_X self-conjugate under the linear non-homogeneous group $LG^{p^n}_{p^n\Omega(p^n)}$ of degree p^n and of order $p^n\Omega(p^n)$, whose general substitution $\sigma_{G,K}$ replaces s_X by $s_{X'}$, where

$$(10) \quad X' = GX + K, \quad x'_i = \sum_{j=1}^{j=n} g_{ij}x_j + k_i \quad (|g_{ij}| \neq 0) \quad (i, j = 1, 2, \ldots n).$$

$\sigma_{G,K}$ replaces $s_{K_1}, s_{K_2}, s_{K_3}$ by $s_{K'_1}, s_{K'_2}, s_{K'_3}$, where

$$K'_1 = GK_1 + K, \quad K'_2 = GK_2 + K, \quad K'_3 = GK_3 + K,$$

so that

$$K'_1 + K'_2 - K'_3 = G(K_1 + K_2 - K_3) + K;$$

hence under $\sigma_{G,K}$ of the $LG^{p^n}_{p^n\Omega(p^n)}$ (10) a multiplication equation of the G_{p^n} $s_{K_1}s_{K_2} = s_{K_3} = s_{K_1+K_2}$ (4, 5) is preserved, that is,

$$s_{K'_1}s_{K'_2} = s_{K'_3} = s_{K'_1+K'_2},$$

if and only if $K = (0) = (k_1, k_2, \ldots k_n) = (0, 0, \ldots 0)$, that is, if and only if the substitution $\sigma_{G,K}$ of the $LG^{p^n}_{p^n\Omega(p^n)}$ (10) is a substitution $\sigma_{G,0} = \sigma_G$ of the $LHG^{p^n}_{\Omega(p^n)}$ (6). We have then this (second) identification of the $\Gamma^{p^n}_{\Omega(p^n)}$ of the Abelian G_{p^n} with the $LHG^{p^n}_{\Omega(p^n)}$.

The group $\Gamma^{p^n}_{\Omega(p^n)} \equiv LHG^{p^n}_{\Omega(p^n)}$ (6) is transitive on the $p^n - 1$ letters $s_X(X \neq (0))$. For $p = 2$ it is doubly transitive on the $p^n - 1 = 2^n - 1$ letters. For $p > 2$ it is simply transitive and imprimitive; the letter $s_X = s_{x_1, x_2, \ldots x_n}$ belongs to and by the ratios of its n suffixes $\underset{\cdot\cdot}{X} = (x_1 : x_2 : \cdots : x_n)$ determines the system of imprimitivity containing the $q-1$ letters* $s_{lX}(l = 1, 2, \cdots \overline{p-1})$; in the G_{p^n} the elements s_{lX} and the identity $s_{0X} = s_{(0)}$ constitute the cyclic group $G_p\{s_X\}$ determined by $s_{\underset{\cdot\cdot}{X}}$, say the $G_{p,\underset{\cdot\cdot}{X}}$. Thus, the $\Gamma^{p^n}_{\Omega(p^n)} \equiv LHG^{p^n}_{\Omega(p^n)}$ permutes first the $(p^n-1)/(p-1)\ G_p$ of G_{p^n}, and afterwards fixes the elements within the various groups. The self-conjugate sub-group which keeps every G_p fixed is of order $p-1$:

---

* $X = (x_1, \cdots x_n)$, $lX = (lx_1, \cdots lx_n)$.

$$(11) \qquad \{X' = lX, \ \ x'_i = lx_i\,(i = 1, 2, \cdot\cdot n)\}, \qquad (l = 1, 2, \cdot\cdot \overline{p-1}).$$

The quotient-group, which is a substitution-group on the $(p^n-1)/(p-1)\ G_p$, has the order $\Omega(p^n)/(p-1)$. Analytically, it is the $LHG^{p^n}_{\Omega(p^n)}$ taken fractionally; that is, the linear fractional group† $LFG^{(p^n-1)/(p-1)}_{\Omega(p^n)/(p-1)}$, whose general substitution $\sigma_{\underset{\cdot\cdot}{G}}$ replaces the $G_{p,\underset{\cdot\cdot}{X}}$ by the $G_{p,\underset{\cdot\cdot}{X}'}$, where

$$(12) \qquad \underset{\cdot\cdot}{X}' = \underset{\cdot\cdot}{G}\underset{\cdot\cdot}{X},^*$$

$$x'_1 : x'_2 : \cdot\cdot : x'_i : \cdot\cdot : x'_n = \sum_{j=1}^{j=n} g_{1j}x_j : \sum_{j=1}^{j=n} g_{2j}x_j : \cdot\cdot : \sum_{j=1}^{j=n} g_{ij}x_j : \cdot\cdot : \sum_{j=1}^{j=n} g_{nj}x_j$$

§ 2.

Three tactical configurations:

$$LCf[p^n], \ \ LHCf[p^n-1], \ \ LFCf[(p^n-1)/(p-1)]:$$

connected with the Abelian G_{p^n} are defining invariants respectively for the three linear groups:

$$LG^{p^n}_{p^n\Omega(p^n)}, \ \ LHG^{p^n \text{ or } p^n-1}_{\Omega(p^n)}, \ \ LFG^{(p^n-1)/(p-1)}_{\Omega(p^n)/(p-1)}$$

The notion *configuration* I transfer to tactic from geometry†; for the proof and ultimate statement of the theorems about to be stated with utmost brevity, this notion must be used to its full content; to-day, however, the term *tactical configuration* shall be merely a name.

The linear configuration $LCf[p^n]$ in p^n letters.

The p^n letters of the $LCf[p^n]$ are the p^n elements s_X of the Abelian G_{p^n}. The G_{p^n} contains $(p^n-1)/(p-1)$ sub-groups,

---

[*] JORDAN: loc. cit., p. 228. In my notation the two subscript dots (..) are the ratio dots (:), and are to call to mind that we may without changing anything replace $X = (x_1, \cdot\cdot x_n)$, $X' = (x'_1, \cdot\cdot x'_n)$, $G = (g_{ij})$ by $lX = (lx_1, \cdot\cdot lx_n)$, $l'X' = (l'x'_1, \cdot\cdot l'x'_n)$, $mG = (mg_{ij})$, respectively, where l, l', m are any integers taken modulo p, but $l \neq 0$, $l' \neq 0$, $m \neq 0$.

[†] See, for instance, REYE: *Das Problem der Configurationen* (*Acta Mathematica*, vol. 1, pp. 92–96, 1882).

$G_{p^{n-1}}$. With respect to each sub-group $G_{p^{n-1}}$ the p^n elements s_X of the G_{p^n} are exhibited as a certain rectangular array of p lines with p^{n-1} elements in each line; the order of the lines and the order of the elements in each line are immaterial; one line contains the p^{n-1} elements of the $G_{p^{n-1}}$ itself. We separate every array into its constituent lines, and have before us in the system of (unordered) $p(p^n-1)/(p-1)$ lines or combinations of p^{n-1} letters each the linear configuration in p^n letters, $LCf[p^n]$.

This $LCf[p^n]$ for $n \geqq 2$ defines, as the maximum substitution-group on the p^n letters s_X leaving it invariant, exactly the $LG^{p^n}_{p^n\Omega(p^n)}$ (§ 1 (10)).

The linear homogeneous configuration $LHCf[p^n-1]$ *in* p^n-1 *letters.*

The p^n-1 letters of the $LHCf[p^n-1]$ are the p^n-1 elements $s_X(X \neq (0))$ of the Abelian G_{p^n}, the identity $s_{(0)}$ excepted. The $LHCf[p^n-1]$ is obtained from the $LCf[p^n]$ by omitting every line or combination containing the discarded letter $s_{(0)}$. The $LHCf[p^n-1]$ consists, then, of a system of p^n-1 lines or combinations of p^{n-1} letters each. This $LHCf[p^n-1]$ is tactically self-reciprocal,[*] that is, we can distribute a notation s'_X to the p^n-1 lines in such a way that the $LHCf[p^n-1]$ on the p^n-1 letters s_X as grouped by the p^n-1 lines s'_X differs only in the priming (′) from the $LHCf[p^n-1]$ on the p^n-1 lines s'_X as grouped by the p^n-1 letters s_X.

This $LHCf[p^n-1]$ for $n \geqq 2$ serves as a defining invariant for exactly the $LHG^{p^n \text{ or } p^n-1}_{\Omega(p^n)}$ (§ 1, (6)). The self-reciprocity of the $LHCf[p^n-1]$ establishes an holoedric isomorphism of the $LHG^{p^n-1}_{\Omega(p^n)}$ with itself. This isomorphism is (at least for $n \geqq 3$) *not*[*] that arising from a transformation of the $LHG^{p^n-1}_{\Omega(p^n)}$ through one of its own elements.

The linear fractional configuration $LFCf[(p^n-1)/(p-1)]$ *on* $(p^n-1)/(p-1)$ *letters.*

The $(p^n-1)/(p-1)$ letters of the $LFCf[(p^n-1)/(p-1)]$ are the $(p^n-1)/(p-1)$ cyclic groups $G_{q;\,\ddot{X}}$ of the Abelian G_{p^n}. The $LFCf[(p^n-1)/(p-1)]$ is obtained from the $LCf[p^n]$ by omitting every line not containing the identity letter $s_{(0)}$, that is, by retaining the lines corresponding to the $(p^n-1)/(p-1)$ sub-groups* $G_{p^{n-1}}$, and then in every such line by omitting the $s_{(0)}$ and replacing every set of $p-1$ letters s_{lX} $(l=1, 2, \ldots q-1)$ by the letter $G_{q;\,X}$. The $LFCf[(p^n-1)/(p-1)]$ on

---

[*] Notice the particular case ($q=2$, $n=3$) in § 2 of my paper cited above. The $LHCf[2^3-1=7]$ and the Δ_7 are, so to say, complementary. Indeed, for $q=2$, $n=$ any, the $LHCf[2^n-1]$ determines uniquely a Δ_{2^n-1}, from which the $LHCf[2^n-1]$ is likewise uniquely determined. This Δ_{2^n-1} serves as a defining invariant for the $LHG^{2^n-1}_{\Omega(2^n)}$.

$$(p^n-1)/(p-1)$$

letters consists then of a system of $(p^n-1)/(p-1)$ lines of $(p^{n-1}-1)/(p-1)$ letters each. This $LFCf[(p^n-1)/(p-1)]$ is tactically self-reciprocal.

This $LFCf[(p^n-1)/(p-1)]$ for $n \geqq 3$ serves as a defining invariant for exactly the $LFG^{(p^n-1)/(p-1)}_{\Omega(p^n)/(p-1)}$ (§ 1, (12)). The self-reciprocity of the $LFCf[(p^n-1)/(p-1)]$ $(n \geqq 3)$ establishes an holoedric isomorphism of the $LFG^{(p^n-1)/(p-1)}_{\Omega(p^n)/(p-1)}$ $(n \geqq 3)$ with itself. This isomorphism is *not* that arising from a transformation of the $LFG^{(p^n-1)/(p-1)}_{\Omega(p^n)/(p-1)}$ through one of its own elements.

In § 4 I give these various tactical configurations for certain low values of p and n.

§ 3.

Utility of the Galois-field theory in the investigation of linear groups.

The results given in § 2 depend for their proof largely upon the fact that *the group* $\Gamma^{p^n-1}_{\Omega(p^n)} \equiv LHG^{p^n-1}_{\Omega(p^n)}$ *contains a substitution* σ_G *which permutes the* p^n-1 *letters* $s_X(X \neq (0))$ *in one cycle of period* p^n-1. Any such σ_G answers the purpose. That one such exists we know from the Galois-field theory.†

---

* This linear fractional configuration might also be called the *sub-group configuration of the Abelian* G_{p^n}.

† Galois: *Sur la théorie des nombres* (*Bulletin des Sciences Mathématiques* de M. Ferussac, vol. 13, p. 428, 1830; reprinted, *Journal de Mathématiques pures et appliquées*, vol. 11, pp. 398–407, 1846.)

Serret: *Algèbre supérieure*, fifth edition, vol. 2, pp. 122–189.

Jordan: *Traité des substitutions*, pp. 14–18.

Moore: *A doubly infinite system of simple groups* (§ 3 is an abstract formulation of the Galois-field theory). (*Proceedings of the Chicago Congress of Mathematics;* in abstract, *Bulletin of the New York Mathematical Society*, vol. 3, Dec. 1893.)

Addendum of Oct. 15, 1895. I have found within a week that Mathieu in Chapter III, pp. 275–304, of his *Mémoire sur l'étude des fonctions de plusieurs quantités, sur la manière de les former et sur les substitutions qui les laissent invariables* (*Journal de Mathématiques pures et appliquées*, ser. 2, vol. 6, pp. 241–323, 1861), working from the Galois-field standpoint, defines and investigates two substitution-groups, which are (otherwise expressed) the groups $LG^{p^n}_{p^n\Omega(p^n)}$ and $LHG^{p^n}_{\Omega(p^n)}$. This seems to be the source from which Mr. Jordan's linear groups (1870) were drawn. Mathieu gives two rational integral functions of the p^n

Suppose that the p^n marks ξ of the Galois-field $GF[p^n]$ of order p^n are exhibited explicitly in terms of n linearly independent marks $\eta_1, \eta_2, \cdots \eta_n$ in the form

$$(1) \qquad \xi = \sum_{i=1}^{i=n} x_i \eta_i,$$

where the x_i's are integral marks, or integers taken modulo p. We make through the $X = (x_1, \cdots x_n)$ a 1.1 correspondence between the letters s_X and the marks ξ. In fact the $GF[p^n]$ quâ additive-group is a concrete Abelian G_{p^n}. Now in the $GF[p^n]$ additions are invariant under the multiplication-substitution σ_γ on the $p^n - 1$ marks ξ $(\xi \neq 0)$,

$$(2) \qquad \xi' = \gamma\xi, \ (\gamma \neq 0),$$

that is, when every mark of the field is multiplied by the same mark γ. Hence this σ_γ interpreted on the s_X is a substitution σ_G of the $\Gamma^{p^n-1}_{\Omega(p^n)} \equiv LHG^{p^n-1}_{\Omega(p^n)}$. If γ is a primitive root of the $GF[p^n]$, σ_γ permutes the $p^n - 1$ marks $\xi (\xi \neq 0)$ in one cycle, and, similarly, σ_G permutes the $p^n - 1$ letters $s_X(X \neq (0))$ in one cycle, and is then the substitution sought.

The results of § 2 constitute for the linear groups sweeping generalizations of Mr. *Noether's* definition* of the group Γ^7_{168} by the triple system Δ_7 in seven letters.

§ 4.

Tables†§ *of the tactical configurations:*
$LCf[p^n]$, $LHCf[p^n - 1]$, $LFCf[(p^n - 1)/(p - 1)]$,
for cases $p^n = 2^2, 3^2, 5^2, 7^2, 11^2; 2^3, 3^3, 5^3, 7^3; 2^4, 3^4, 5^4; 2^5; 2^6$.

The table for a particular case $[p^n]$ gives first a primitive root γ of the Galois-field $GF[p^n]$ and its fundamental equation

---

letters s_X, each of which serves as defining invariant for the $LG^{p^n}_{p^n(p\Omega^n)}$. These functions are closely related to our $LCf[p^n]$. In explaining my researches in detail in a subsequent paper I shall point out the exact points of contact with MATHIEU's results.

It should be added that several weeks ago Mr. Dickson and I came upon a substitution-group on the p^n marks of the $GF[p^n]$ which Mr. Dickson then identified as another expression of Mr. Jordan's $LHG^{p^n}_{\Omega(p^n)}$; this was exactly MATHIEU's expression of the group.

* See § 2 of my paper cited above.

† The theory of the linear fractional configuration I introduced in my course *Groups*, during the last spring quarter at the University of Chicago, and in connection with the members of that course, Messrs. Brown, Dickson, Joffe, and Slaught, worked up the linear fractional configurations § for the cases given above, except $p^n = 2^6$. I take this opportunity to thank them for their coöperation, and especially Mr. Dickson, who quite recently completed the tables as given above.

§ I add the tables for $p^n = 2^2, 3^2, 5^2, 7^2, 11^2$, whose linear *fractional* configurations are trivial. Sept. 10, 1895.

of degree n. The p^n elements of the abstract G_{p^n} have the *index-notation*[*] derived from the p^n marks of the $GF[p^n]$ (as a concrete G_{p^n}, § 3): mark $\xi = 0$, index $*$; mark $\xi \neq 0$, $\xi = \gamma^i$, index i $(i = 0, 1, \ldots \overline{p^n - 1})$; i is an integer taken modulo $p^n - 1$.

The $LCf[p^n]$ consists (§ 2) of the lines found in certain $(p^n - 1)/(p - 1)$ arrays. Each array has p lines; each line has p^{n-1} indices. Only the first array is given; the others are obtained from it by repeated applications of the cyclical substitution

$$i' = i + 1, \qquad (i = 0, 1, \cdots \overline{p^n - 1}),$$

which leaves $*$ fixed. The first line of the first array is the additive sub-group $G_{p^{n-1}}$ of the $GF[p^n]$ quâ additive G_{p^n}, which contains the $n - 1$ marks $\gamma^0, \gamma^1, \cdots \gamma^{n-2}$. The second line is obtained by adding the *mark* γ^{n-1} to the *marks* of the first line. Of course the lines one and two must be expressed in the index-notation. The following lines are derived from the second at once by repeated additions of $(p^n - 1)/(p - 1)$ to the *indices* of the second line.

The $LHCf[p^n - 1]$ and the $LFCf[(p^n - 1)/(p - 1)]$ are easily derived from the $LCf[p^n]$ (§ 2). The $LCf[p^n]$ and the $LHCf[p^n - 1]$ are tabulated together.

TABLES.[†]

$[p^n = 2^2]$	$GF[2^2]$	Primitive root γ where $\gamma^2 = 1 + \gamma$.
	$LCf[2^2]$	4 indices $*$ 0 1 2.
	$LHCf[2^2 - 1]$	3 indices 0 1 2.

$[*\ 0]_2 \quad [1\ 2]_2$ [§]

$[p^n = 3^2]$	$GF[3^2]$	Primitive root γ where $\gamma^2 = 1 + 2\gamma$.
	$LCf[3^2]$	9 indices $*$ 0 1 . . 7.
	$LHCf[3^2 - 1]$	8 indices 0 1 . . 7.

$[*\ 0\ 4]_3 \quad [1\ 6\ 7]_3 \quad [2\ 3\ 5]_3$

$[p^n = 5^2]$	$GF[5^2]$	Primitive root γ where $\gamma^2 = 2 + 2\gamma$.
	$LCf[5^2]$	25 indices $*$ 0 1 . . 23.
	$LHCf[5^2 - 1]$	24 indices 0 1 . . 23.

$[*\ 0\ 6\ 12\ 18]_5 \quad [1\ 3\ 4\ 8\ 17]_5 \quad [7\ 9\ 10\ 14\ 23]_5 \quad [5\ 13\ 15\ 16\ 20]_5$
$[2\ 11\ 19\ 21\ 22]_5$

---

[*] The s_x notation for the elements can be recovered if necessary.

[†] The $LFCf\left[\dfrac{p^n - 1}{p - 1}\right]$ for $n = 2$ is trivial and hence is not tabulated.

[§] Every *line* [ ] has a suffix indicating the number of *indices* lying within.

$[p^n = 7^2]$	$GF[7^2]$	Primitive root γ where $\gamma^2 = 2 + 2\gamma$.
	$LCf[7^2]$	49 indices $*$ 0 1 . . 47.
	$LHCf[7^2 - 1]$	48 indices 0 1 . . 47.

$[*\ 0\ 8\ 16\ 24\ 32\ 40]_7$

$[1\ 6\ 18\ 20\ 21\ 27\ 31]_7$ $[9\ 14\ 26\ 28\ 29\ 35\ 39]_7$ $[17\ 22\ 34\ 36\ 37\ 43\ 47]_7$

$[3\ 7\ 25\ 30\ 42\ 44\ 45]_7$ $[2\ 4\ 5\ 11\ 15\ 33\ 38]_7$ $[10\ 12\ 13\ 19\ 23\ 41\ 46]_7$

$[p^n = 11^2]$	$GF[11^2]$	Primitive root γ where $\gamma^2 = 9 + 4\gamma$.
	$LCf[11^2]$	121 indices $*$ 0 1 . . 119.
	$LHCf[11^2 - 1]$	120 indices 0 1 . . 119.

$[*\ 0\ 12\ 24\ 36\ 48\ 60\ 72\ 84\ 96\ 108]_{11}$

$[1\ 27\ 55\ 58\ 65\ 66\ 71\ 80\ 98\ 100\ 117]_{11}$

[Second line] + 12; 24; 36; 48; 60; 72; 84; 96; 108 = the respective remaining lines.

$[p^n = 2^3]$	$GF[2^3]$	Primitive root γ where $\gamma^3 = 1 + \gamma$.
	$LCf[2^3]$	8 indices $*$, 0, 1, . . 7.
	$LHCf[2^3 - 1]$	7 indices 0, 1, . . 7.

$[*\ 0\ 1\ 3]_4$ $[2\ 4\ 5\ 6]_4$

$LFCf[(2^3 - 1)/(2 - 1)]$ 7 indices 0, 1, . . 7.

$[0\ 1\ 3]_3$

$[p^n = 3^3]$	$GF[3^3]$	Primitive root γ where $\gamma^3 = 2 + \gamma$.
	$LCf[3^3]$	27 indices $*$, 0, 1, . . 25.
	$LHCf[3^3 - 1]$	26 indices 0, 1, . . 25.

$[*\ 0\ 1\ 3\ 9\ 13\ 14\ 16\ 22]_9$

$[2\ 4\ 6\ 7\ 10\ 11\ 12\ 18\ 21]_9$

$[15\ 17\ 19\ 20\ 23\ 24\ 25\ 5\ 8]_9$

$LFCf[(3^3 - 1)/(3 - 1)]$ 13 indices 0, 1, . . 12.

$[0\ 1\ 3\ 9]_4$

$[p^n = 5^3]$	$GF[5^3]$	Primitive root γ where $\gamma^3 = 3 + 2\gamma$.
	$LCf[5^3]$	125 indices $*$, 0, 1, . . 123.
	$LHCf[5^3 - 1]$	124 indices 0, 1, . . 123.

$[*\ 0\ 1\ 3\ 10\ 14\ 26\ 31\ 32\ 34\ 41\ 45\ 57\ 62\ 63\ 65\ 72\ 76\ 88\ 93\ 94\ 96\ 103\ 107\ 119]_{25}$

$[2\ 9\ 13\ 15\ 28\ 29\ 30\ 35\ 38\ 39\ 48\ 53\ 56\ 68\ 80\ 82\ 98\ 104\ 105\ 109\ 112\ 114\ 116\ 117\ 120]_{25}$

[Second line] + 31; 62; 93 = the respective remaining lines.

$LFCf[(5^3 - 1)/(5 - 1)]$ 31 indices 0, 1, . . 30.

$[0\ 1\ 3\ 10\ 14\ 26]_6$

$[p^n = 7^3]$ $GF[7^3]$ Primitive root γ where $\gamma^3 = 5 + \gamma$.
$LCf[7^3]$ 343 indices *, 0, 1, . . 341.
$LHCf[7^3 - 1]$ 342 indices 0, 1, . . 341.

[* 0 1 3 13 32 36 43 52 57 58 60 70 89 93 100 109 114 115 117 127 146 150 157 166 171 172 174 184 203 207 214 223 228 229 231 241 260 264 271 280 285 286 288 298 317 321 328 337]$_{49}$

[2 4 6 9 14 16 26 33 35 41 44 45 46 50 56 64 75 78 82 86 99 133 134 142 148 168 181 186 194 195 201 202 218 219 222 240 245 265 267 268 277 281 283 290 293 296 307 312 323]$_{49}$

[Second line] + 57 ; 114 ; 171 ; 228 ; 285 = the respective remaining lines.

$LFCf[(7^3 - 1)/(7 - 1)]$ 57 indices 0, 1, . . 56.

[0 1 3 13 32 36 43 52]$_8$

$[p^n = 2^4]$ $GF[2^4]$ Primitive root γ where $\gamma^4 = 1 + \gamma$.
$LCf[2^4]$ 16 indices *, 0, 1, . . 14.
$LHCf[2^4 - 1]$ 15 indices 0, 1, . . 14.

[* 0 1 2 4 5 8 10]$_8$ [3 6 7 9 11 12 13 14]$_8$

$LFCf[(2^4 - 1)/(2 - 1)]$ 15 indices 0, 1, . . 14.

[0 1 2 4 5 8 10]$_7$

$[p^n = 3^4]$ $GF[3^4]$ Primitive root γ where $\gamma^4 = 1 + \gamma + 2\gamma^2 + 2\gamma^3$.
$LCf[3^4]$ 81 indices *, 0, 1, . . 79.
$LHCf[3^4 - 1]$ 80 indices 0, 1, . . 79.

[* 0 1 2 5 12 18 22 24 26 27 29 32 33 40 41 42 45 52 58 62 64 66 67 69 72 73]$_{27}$

[3 7 15 17 20 21 30 31 37 38 44 46 48 49 50 51 53 54 56 59 63 65 68 74 75 76 79]$_{27}$

[Second line] + 40.

$LFCf[(3^4 - 1)/(3 - 1)]$ 40 indices 0, 1, . . 39.

[0 1 2 5 12 18 22 24 26 27 29 32 33]$_{18}$

$[p^n = 5^4]$ $GF[5^4]$ Primitive root γ where $\gamma^4 = 2 + \gamma + \gamma^2$.
$LCf[5^4]$ 625 indices *, 0, 1, . . 623.
$LHCf[5^4 - 1]$ 624 indices 0, 1, . . 623.

[* {0 1 2 7 18 19 23 36 43 44 46 47 55 57 61 64 70 76 77 84 86 89 92 94 96 108 119 122 143 148 152} + 0, 156, 312, 468]$_{125}$

[3 4 5 10 17 21 22 29 31 37 39 41 42 59 63 68 74 88 95 99 104 107 109 110 127 130 134 141 146 153 162 165 168 169 181 186 189 190 191 194 196 207 208 216 218 221 222 225 229 231 237 239 241 261 262 269 270 271 272 276 277 279 281 285 287 288 289 291 294 300 305 306 323 328 332 336 338 361 362 365 366 379 383 390 399 402 405 410 412 413 415 424 430 438 440 451 454 467 476 482 483 495 496 500 513 516 524 526 540 547 548 550 559 565 570 579 585 592 604 605 608 613 615 619 622]$_{125}$

[Second line] + 156 ; 312 ; 468 = the respective remaining lines.

$LFCf[(5^4 - 1)/(5 - 1)]$ 156 indices 0, 1, . . 155.

[The { } of first line above]$_{31}$

$[p^n = 2^5]$ *GF*$[2^5]$ Primitive root γ where $\gamma^5 = 1 + \gamma + \gamma^2 + \gamma^3$.

LCf$[2^5]$ 32 indices $*$, 0, 1, . . 30.

LHCf$[2^5 - 1]$ 31 indices 0, 1, . . 30.

$[* \; 0 \; 1 \; 2 \; 3 \; 5 \; 8 \; 10 \; 12 \; 13 \; 14 \; 18 \; 24 \; 25 \; 27 \; 28]_{16}$

$[4 \; 6 \; 7 \; 9 \; 11 \; 15 \; 16 \; 17 \; 19 \; 20 \; 21 \; 22 \; 23 \; 26 \; 29 \; 30]_{16}$

LFCf$[(2^5 - 1)/(2 - 1)]$ 31 indices 0, 1, . . 30.

$[0 \; 1 \; 2 \; 3 \; 5 \; 8 \; 10 \; 12 \; 13 \; 14 \; 18 \; 24 \; 25 \; 27 \; 28]_{15}$

$[p^n = 2^6]$ *GF*$[2^6]$ Primitive root γ where $\gamma^6 = 1 + \gamma + \gamma^3 + \gamma^4$.

LCf$[2^6]$ 64 indices $*$, 0, 1, . . 62.

LHCf$[2^6 - 1]$ 63 indices 0, 1, . . 62.

$[* \; 0 \; 1 \; 2 \; 3 \; 4 \; 6 \; 13 \; 14 \; 16 \; 18 \; 20 \; 21 \; 22 \; 25 \; 26 \; 31 \; 35 \; 37 \; 40 \; 42 \; 43 \; 46 \; 49$
$50 \; 51 \; 53 \; 54 \; 56 \; 57 \; 58 \; 59]_{32}$

$[5 \; 7 \; 8 \; 9 \; 10 \; 11 \; 12 \; 15 \; 17 \; 19 \; 23 \; 24 \; 27 \; 28 \; 29 \; 30 \; 32 \; 33 \; 34 \; 36 \; 38 \; 39 \; 41$
$44 \; 45 \; 47 \; 48 \; 52 \; 55 \; 60 \; 61 \; 62]_{32}$

LFCf$[(2^6 - 1)/(2 - 1)]$ 63 indices 0, 1, . . 62.

[First line above, omitting the $*]_{31}$

THE UNIVERSITY OF CHICAGO,
August 25, 1895.

ELEMENTARY PROOF OF THE QUATERNION ASSOCIATIVE PRINCIPLE.[*]

BY PROFESSOR ARTHUR S. HATHAWAY.

THE variety of demonstrations that Hamilton has given of the associative principle of quaternion multiplication, and the remarks that he has made upon such demonstrations, show that he considered an elementary proof of this principle as very desirable. Only two of Hamilton's proofs have been generally employed by subsequent writers—the direct proof by spherical conics, and the indirect one depending upon the assumed laws of i, j, k—and the proof that he considered the most elementary has been entirely ignored, probably because of its deviation from fundamental ideas. On page 297 of the *Elements*, Hamilton calls attention to another method, as follows: "The *associative principle* of multiplication may also be proved without the distributive principle, by certain considerations of *rotations of a system*, on which we cannot enter here."

It is, of course, easy to see that such a proof is possible; but the details of it could not have presented themselves to Hamilton in an elementary form, or he would have seen that it was just the demonstration for which he was looking, simple in character, and direct in its application.

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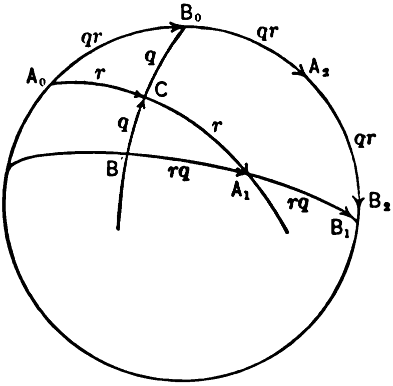
[*] Presented to the AMERICAN MATHEMATICAL SOCIETY August 28, 1895.

The foundation consists of a few simple data with regard to the rotation of a sphere about its diameters, and a lemma that immediately identifies versor multiplication with the composition of rotations.

By the rotation, 2 *arc* q, we mean the rotation whose *equator, angular magnitude*, and *direction*, are given by double the arc that gives the *plane, angular magnitude*, and *direction*, of the turning or versor operation of q. The axis of q is therefore the axis of the rotation; and as *arc* q is not greater than a semicircumference, so 2 *arc* q is not greater than a circumference. There is not more than one axis around which the sphere may be rotated from one position to another, since any diameter that is oblique to one axis returns to its original position only for entire revolutions of the sphere, and cannot therefore be another axis of rotation. Hence there is not more than one equator pertaining to the two positions of the sphere, and all arcs that specify rotations between those positions must lie upon the same great circle and be equal in magnitude and direction to one or the other of two oppositely directed arcs whose combined lengths make a circumference. The halves of any two such arcs are therefore either equal and concurrent, or supplementary and opposite.

Lemma. The rotation 2 arc r*, followed by the rotation 2 arc* q*, produces the same displacement of the sphere as the single rotation 2 arc* qr.



For let C be a point of intersection of the great circles of r and q, and lay off the given arcs, *arc* $r = A_0C = CA_1$, *arc* $q = B'C = CB_0$; and also construct *arc* $rq = B'A_1 = A_1B_1$, and *arc* $qr = A_0B_0 = B_0A_2 = A_2B_2$. By construction, the spherical triangles A_0B_0C, $A_1B'C$, are equal; and therefore the arcs A_0B_0, A_1B_1, are equal and equally inclined to the arc A_0A_1, and the continuous arcs $B'A_1$, A_1B_1, are equal to the continuous arcs B_0A_2, A_2B_2, and these two continuous arcs are equally inclined to $B'B_0$. Hence, if a marked arc AB of the sphere initially coincide with A_0B_0, then the rotation of the sphere through $2\ arc\ r = A_0A_1$, brings AB into the position A_1B_1, and the next rotation through $2\ arc\ q = B'B_0$, brings AB into the position A_2B_2. From the figure it is obvious that the position A_2B_2 is reached from the position A_0B_0 by the single rotation $2\ arc\ qr = A_0A_2$. Q.E.D.

This lemma shows that the successive rotations, $2\ arc\ r$, $2\ arc\ q$, $2\ arc\ p$, produce the same displacement of the sphere as either of the single rotations, $2\ arc\ p(qr)$, or $2\,arc(pq)r$ (according as we begin by compounding the first and second, or the second and third, rotations into their equivalent single rotations). Hence, $arc\ p(qr)$ and $arc(pq)r$ are equal and concurrent, or supplementary and opposite arcs of the same great circle. We may rely upon a diagram to show the currency of these two arcs, and therefore their complete versor equality, or we may prove the same by the principles of continuity, as follows: When p, q, r, change continuously in value, their arcs and the arcs of their products also change continuously in circle length and direction. Hence, if $arc\ p(qr)$ and $arc\ (pq)r$ are equal and concurrent for particular values of p, q, r (as is obviously the case), they must continue so for all values of p, q, r, since an indefinitely small change in each arc cannot change them from equal concurrent arcs to the only other possible case of supplementary opposite arcs.

So in general, if the sphere receive the successive rotations, $2\,arc\ r$, $2\,arc\ q$, $2\,arc\ p$, . . . , then its displacement is the same as by the single rotation, $2\,arc\ (\ldots pqr)$, in whatever way the factors of this product are associated to form such product. Therefore the arcs of any two such products are either equal and concurrent, or supplementary and opposite arcs of the same great circle; and the principles of continuity, and the special case in which each arc begins where the preceding arc ends, show that the first-named relation is the true one. Since the tensors of all such products are also equal, therefore all such products, formed by different associations of the same factors in the same order, are equal.

Rose Polytechnic Institute,
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MORAL VALUES.*

BY R. HENDERSON.

THE question of moral values in connection with the Theory of Probability is one which has given rise to great diversity of opinion among mathematicians. In Bertrand's *Calcul des Probabilités*, which is, perhaps, the most important work of recent years on the subject of Probability, it is dismissed with contempt, while in some other works considerable space is devoted to it.

Perhaps one of the reasons for the disrepute in which some hold the theory of moral values, is the fact that almost all who have taken up the subject from a mathematical standpoint have adhered to Bernoulli's or some other equally rigid hypothesis. Bernoulli's hypothesis is that the increase in moral value, arising from an indefinitely small increase in wealth, varies directly as that increase and inversely as the wealth already in possession, or, in other words, denoting the moral value of a wealth x by the symbol $f(x)$, that $\frac{df(x)}{dx} = \frac{a}{x}$, where a is a constant, or, integrating this, that $f(x) = a \log x + c$, c being another arbitrary constant. The form of $f(x)$ adopted by Cramer was $f(x) = ax^{\frac{1}{2}}$. It would seem that many, being unwilling to admit these hypotheses, have been inclined to ridicule the whole subject, and have been to a certain extent blind to the basis of fact on which it is founded.

Now it has long seemed to me that Bernoulli did not intend his hypothesis as an absolute statement of the law of moral values, but rather as merely a particular case of a more general hypothesis, however definite or indefinite in his mind that general hypothesis may have been. And indeed writers on the subject of Economics, to whose department the question of the admissibility, or the contrary, of the hypothesis properly belongs, have adopted, though I cannot say they have originated, a more general hypothesis treated of below.

The earliest statement of this general hypothesis which I have seen is in the introduction to Laplace's *Théorie Analytique des Probabilités*, page xxii of the edition of 1847, at which place there occurs a passage of which the following is a free translation:

"The disadvantage of gambling and of exposing to the same risk all goods to be shipped, and all similar results indicated by good sense, subsist, whatever function the moral value may be of the material fortune of each individual. It is sufficient that the ratio of the increase of that function to the increase of the material fortune diminish as this latter increases."

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* Presented to the AMERICAN MATHEMATICAL SOCIETY, October 26, 1895.

Professor Marshall in his *Principles of Economics*, Appendix, Note ix, by a course of reasoning similar to that below, proves the disadvantage of gambling, provided $f(x)$ is amenable to expansion in Taylor's series in the form which he uses, that is, provided $\frac{d^2f(x)}{dx^2}$ is continuous and finite; but I have not been able to find anywhere a complete proof of the truth of these statements of Laplace, and it might be interesting to examine them analytically.

Denoting, as above, the moral value of a material fortune of amount x, by $f(x)$, Laplace's statement is equivalent to saying that it is sufficient that $\frac{df(x)}{dx}$ should diminish as x increases, or, in other words, that $\frac{d^2f(x)}{dx^2}$ should be always negative, but not necessarily finite nor continuous. I shall introduce the extra assumption that $\frac{df(x)}{dx}$ is always finite which enables me to use the following expression for $f(x+h)$, the source of which I cannot at this moment recollect.

$$\begin{aligned} f(x+h) &= f(x) + \int_0^h f'(x+h-z)\,dz \\ &= f(x) + hf'(x) + \int_0^h zf''(x+h-z)\,dz \\ &= f(x) + hf'(x) + \theta h\int_0^h f''(x+h-z)\,dz \\ &= f(x) + hf'(x) + \theta h\,\{f'(x+h) - f'(x)\}, \end{aligned}$$

where θ is a proper fraction, since $f''(x+h-z)$ retains the same sign throughout. And since that sign is negative, the sign of $\int_0^h f''(x+h-z)\,dz$ is necessarily different from that of h, and consequently the last term $\theta h\{f'(x+h)-f'(x)\}$ is essentially negative. These results hold subject to the condition above stated that $f'(x)$ should be always finite.

This equation then may be expressed in the form

$$f(x+h) < f(x) + hf'(x)$$

for all values of h. From this inequality it is easy to deduce the following theorem:

Theorem. If $f(x)$ be a function of x such that $f'(x)$ is finite and $f''(x)$ is negative for all values of x, then for all positive values of l and m we have

$$\frac{lf(a)+mf(b)}{l+m} < f\left(\frac{la+mb}{l+m}\right).$$

For, let $\frac{la+mb}{l+m}=x$ and $\frac{b-a}{l+m}=h$,

then $a=x-mh$ and $b=x+lh$,

and
$$\frac{lf(a)+mf(b)}{l+m}=\frac{l}{l+m}f(x-mh)+\frac{m}{l+m}f(x+lh)$$
$$<\frac{l}{l+m}\{f(x)-mhf'(x)\}+\frac{m}{l+m}\{f(x)+lhf'(x)\},$$
$$i.e.\quad <f(x),$$
$$i.e.\quad <f\left(\frac{la+mb}{l+m}\right).$$

Corollary. Applying this theorem successively, we get
$$\frac{lf(a)+mf(b)+nf(c)+\cdots}{l+m+n+\cdots}<f\left(\frac{la+mb+nc+\cdots}{l+m+n+\cdots}\right),$$
where l, m, n, etc., are all positive.

Supposing now that the moral value of a material fortune x may be represented by $f(x)$, a function which satisfies the above conditions, we have the following results:

(1) *A bet at fair odds is morally disadvantageous.*

For, let the man's material fortune before making the bet be a, and let x be staked against y in a fair wager.

Then the probability of winning is $\frac{x}{x+y}$, and of losing is $\frac{y}{x+y}$. Assuming then that the moral value of his expectation is equal to the sum of the moral values of the various amounts of which he may ultimately stand possessed multiplied respectively by their probabilities, we have it equal, after the bet is made, to
$$\frac{x}{x+y}f(a+y)+\frac{y}{x+y}f(a-x),$$
which by the theorem is less than
$$f\left(\frac{x}{x+y}(a+y)+\frac{y}{x+y}(a-x)\right),$$
or $f(a)$ the moral value at first.

(2) *If a merchant expects a shipment, the moral value of his expectation is greater if the shipment be divided than if it be shipped in one vessel only, the probability of each vessel arriving being the same.*

Let v be the merchant's wealth in possession, s the value of the shipment, and p the probability of each vessel arriving, and let $(1-p)=q$.

Then if it be shipped in one vessel, the moral value of his expectation is $pf(v+s)+qf(v)$.

But if it be shipped in t different vessels,

The probability of

all arriving is p^t, when his wealth will be $(v+s)$.

$t-1$ " $tp^{t-1}q$, " $\left(v+\frac{t-1}{t}s\right)$.

$t-2$ " $\frac{t(t-1)}{1\cdot 2}p^{t-2}q^2$, " $\left(v+\frac{t-2}{t}s\right)$.

etc.

Therefore the moral value of his expectation is

$$p^t f(v+s)+tp^{t-1}qf\left(v+\frac{t-1}{t}s\right)+\frac{t(t-1)}{1\cdot 2}f\left(v+\frac{t-2}{t}s\right)$$

$$+\cdots+q^t f(v),$$

$$>p^t f(v+s)+\{(t-1)p^{t-1}qf(v+s)+p^{t-1}qf(v)\}$$

$$+\left\{\frac{(t-1)(t-2)}{1\cdot 2}f(v+s)+\frac{t-1}{1}f(v)\right\}+\cdots+q^t f(v),$$

$$\textit{i.e.}\ >p(p+q)^{t-1}f(v+s)+q(p+q)^{t-1}f(v),$$

$$\textit{i.e.}\ >pf(v+s)+qf(v),$$

which is the moral value of his expectation if the shipment be in one vessel only.

(3) *In the above case, the larger the number of vessels, the greater is the moral value of his expectation.*

For if it be shipped in $(t+1)$ vessels, the moral value of his expectation is

$$p^{t+1}f(v+s)+(t+1)p^t qf\left(v+\frac{t}{t+1}s\right)$$

$$+\frac{(t+1)t}{1\cdot 2}p^{t-1}q^2 f\left(v+\frac{t-1}{t+1}s\right)+\cdots+q^{t+1}f(v).$$

Now $$v+\frac{t}{t+1}s=\frac{1}{t+1}(v+s)+\frac{t}{t+1}\left(v+\frac{t-1}{t}s\right),$$

$$v+\frac{t-1}{t+1}s=\frac{2}{t+1}\left(v+\frac{t-1}{t}s\right)+\frac{t-1}{t+1}\left(v+\frac{t-2}{t}s\right),$$

etc.

Therefore this value is greater than

$$p^{t+1}f(v+s)+\left\{p^tqf(v+s)+tp^tqf\left(v+\frac{t-1}{t}s\right)\right\}$$
$$+\left\{tp^{t-1}q^2f\left(v+\frac{t-1}{t}s\right)+\frac{t(t-1)}{1\cdot 2}p^{t-1}q^2f\left(v+\frac{t-2}{t}s\right)\right\}$$
$$+\cdots+\left\{(tpq^tf\left(v+\frac{1}{t}s\right)+pq^tf(v)\right\}+q^{t+1}f(v)$$
$$=p^tf(v+s)+tp^{t-1}qf\left(v+\frac{t-1}{t}s\right)+\frac{t(t-1)}{1\cdot 2}f\left(v+\frac{t-2}{t}s\right)$$
$$+\cdots+q^tf(v),$$

which is the moral value if it be shipped in vessels.

4. *In the above case the merchant may advantageously pay more than the mathematical net premium to insure the cargo.*

For suppose he merely pays the net premium qs. After paying it his wealth is $(v+ps)$, and its moral value is $f(v+ps)$.

Now taking the general case of t vessels, before insuring, the moral value of his expectation is, as above,

$$p^tf(v+s)+tp^{t-1}qf\left(v+\frac{t-1}{t}s\right)+\frac{t(t-1)}{1\cdot 2}p^{t-2}q^2f\left(v+\frac{t-2}{t}s\right)$$
$$+\cdots+q^tf(v).$$

Applying the corollary, we find this to be less than

$$f\left\{p^t(v+s)+tp^{t-1}q\left(v+\frac{t-1}{t}s\right)+\frac{t(t-1)}{1\cdot 2}p^{t-2}q^2\left(v+\frac{t-2}{t}s\right)\right.$$
$$\left.+\cdots+q^tv\right\}$$
$$=f\{v(p+q)^t+ps(p+q)^{t-1}\}$$
$$=f(v+ps),$$

the moral value after insuring at the net premium. He may therefore without disadvantage pay more than the net premium.

The following diagram illustrates the case where $t=1$.

In this diagram the equation of the curve $PSRQ$ is supposed to be $y=f(x)$. On the axis of x points M, N, K are taken, so that $OM=v$; $MN=s$; and $MK=ps$; and therefore

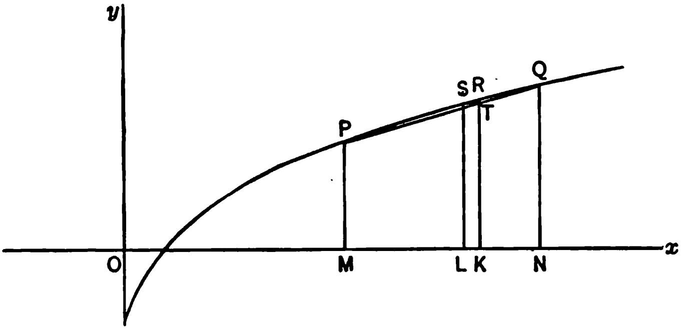
$$ON=v+s \text{ and } OK=v+ps.$$

At M, N, and K ordinates are erected meeting the curve at P, Q, and R, respectively, so that we have

$$MP=f(v),\ NQ=f(v+s), \text{ and } KR=f(v+ps).$$

The chord PQ being drawn meets KR at T, so that

$$KT = pf(v+s) + qf(v).$$



Through T draw TS parallel to the axis of x to meet the curve in S, and through S draw the ordinate SL meeting the axis of x in the point L; then since $SL = TK$, we have

$OL =$ fortune in possession whose moral value equals the moral value of the expectation.
$LN =$ premium which may, without disadvantage, be paid to insure S, represented by MN.
$KN =$ net premium for same.
$LK = ST =$ loading allowed.

OTTAWA, *July* 12, 1895.

NOTES.

A REGULAR meeting of the AMERICAN MATHEMATICAL SOCIETY was held in New York, Saturday afternoon, October 26, at three o'clock, the President, Dr. HILL, in the chair. There were nineteen members present. On the recommendation of the Council, the following persons, nominated at the preceding meeting, were elected to membership: Professor HENRY BYRON NEWSON, University of Kansas, Lawrence, Kas.; and Professor FREDERICK SHENSTONE WOODS, Massachusetts Institute of Technology, Boston, Mass. Seven nominations for membership were received. The President announced to the Society that an invitation had been received from the President of the Northwestern University, Evanston, Ill., inviting the Society to hold its next summer meeting at

Evanston under the auspices of the Northwestern University. He stated that he had replied to the invitation on behalf of the Society, expressing its appreciation of the offer, although it was not possible as yet to make any definite arrangements in regard to a meeting next summer.

The following paper, transmitted to the Society by Mr. J. C. GLASHAN, was presented: "On Moral Values," by Mr. R. HENDERSON. The paper was read by the Secretary. It appears in the present number of the BULLETIN. Professor R. S. WOODWARD communicated a note on an error in Church's Descriptive Geometry relative to the surface generated by the motion of a straight line tangent to a helix.

PROFESSOR FRANZ NEUMANN, well known through his works on mathematical physics, died at Königsberg, on the 23d of May, at the advanced age of 96.

THE director of the Astronomische Recheninstitut and professor of astronomy in the University of Berlin, Dr. F. Tietjen, died at Berlin on the 21st of June, aged 61.

PROFESSOR FRIEDRICH GUSTAV MEHLER, who died in July, was the author of several papers published in *Crelle's Journal* and of a widely used text-book of elementary mathematics (Hauptsätze der Elementar-Mathematik). At the time of his death he was professor in the Elbing Gymnasium.

THE mathematical prize of the Prince Jablonowski Society for 1894 (1000 marks) has been awarded to Dr. Paul Harzer, director of the Gotha Observatory. The problem proposed by the Society required *a new determination of the secular perturbations of at least the orbits of Mercury, Venus, the Earth, and Mars, having regard to the terms of higher order* (see BULLETIN OF THE AMERICAN MATHEMATICAL SOCIETY, Vol. I, p. 22). It appears from the Annual Report of the Prince Jablonowski Society, dated Leipzig, March, 1895, that while Dr. Harzer's memoir contains a complete solution of the theoretical part, the practical part of the problem is carried only through the preliminary calculations up to the actual integration. But as the calculations are carried sufficiently far to warrant a very favorable judgment concerning the adequacy of the method used by the author, the Society has decided to publish the memoir, urgently requesting the author to continue his work so as to effect the numerical evaluation of the secular perturbations.

THE University of the City of New York has appointed Mr. Arthur D. Frizell an assistant in mathematics.

THE organization of an "International Association for Promoting the Study of Quaternions and Allied Systems of Mathematics" has been suggested. Information in regard to the proposed society can be obtained from Messrs. P. Molenbroek, The Hague, Holland, and S. Kimura, Yale University, New Haven.

B. G. TEUBNER of Leipzig has an unusually large number of mathematical works in preparation.

To begin with the history of mathematics, the second part of Vol. III of MORITZ CANTOR's "Vorlesungen über Geschichte der Mathematik," covering the period from 1700 to 1726, is announced. The third part, which is to conclude this standard work, will carry the historical account to the year 1759. The excellent edition, by I. L. Heiberg and H. Menge, of the complete works of Euclid ("EUCLIDIS Opera omnia," Greek text with Latin translation) of which Vols. I–V and Vol. VII have appeared before, will be brought to a close this year by the appearance of Vol. VI, which contains the "Data" edited by Dr. Menge. As the second volume of Paul Tannery's edition of Diophantus ("DIOPHANTI ALEXANDRINI Opera omnia cum Graecis commentariis") is in press, we are now in possession of most carefully prepared texts of the four classics of Greek mathematics: Euclid, Archimedes, Apollonius, and Diophantus. A complete edition of the works of Ptolemy, by Berger, Boll, Heiberg, and others, is in preparation.

In modern mathematics Teubner is also publishing several sets of collected works. The second part of Vol. I of HERMANN GRASSMANN's mathematical writings, edited by Professor F. Engel, contains the "Ausdehnungslehre" of 1862, and will probably be out before the end of the present year. Two more volumes are to follow. The mathematical volume of JULIUS PLÜCKER's "Wissenschaftliche Abhandlungen," edited by Professor A. Schoenflies, has just appeared; the physical volume will be edited by Dr. F. Pockels. The first volume of LEOPOLD KRONECKER's "Vorlesungen," which appeared in 1894, has now been followed by the first volume of his "Gesammelte Werke"; each of these two series will consist of four volumes. P. STÄCKEL and F. ENGEL's "Theorie der Parallellinien von Euklid bis auf Gauss, eine Urkundensammlung," and O. BIERMANN's "Elemente der höheren Mathematik," appeared during the last few months. The following works are announced as in press: a German translation, by M. Lukat, of LUIGI BIANCHI's "Geometria differenziale"; a new work by Professor CARL NEUMANN, "Allgemeine Untersuchungen über das Newton'sche Princip der Fernwirkungen"; the first volume of a "Theorie der doppelt periodischen Funktionen einer Veränderlichen," by Professor MARTIN KRAUSE; the second part of Vol. II, as well as Vol. III, of Professor Ernst Schröder's "Vorlesungen über die Algebra der Logik"; and the long expected "Geometrie der Zahlen," by Professor Hermann Minkowski. The printing of the first part of a new work by Professor Sophus Lie and Dr. Georg Scheffers, "Geometrie der Berührungstransformationen," is nearly completed, and the work will probably appear in December.

Among other works announced as soon forthcoming, we may mention: L. Schlesinger, "Handbuch der Theorie der linearen Differentialgleichungen," Vol. II; O. Stolz, "Grundzüge der Differential- und Integralrechnung," Vol. II; R. Sturm, "Die Gebilde ersten und zweiten Grades der Liniengeometrie in synthetischer Behandlung," Vol. III; P. Gordan, "Vorlesungen über Invariantentheorie," Vol. III; and W. Wirtinger, "Untersuchungen über Thetafunktionen," a memoir which was awarded the Benecke prize by the Philosophical Faculty of the University of Göttingen in 1895. A work on automorphic functions, by Professor Felix Klein and Dr. Robert Fricke, is also in preparation.

In applied mathematics Teubner announces the second volume of the fourth edition of Wüllner's "Experimental-Physik"; Vol. VIII of Franz Neumann's "Vorlesungen über mathematische Physik: Vorlesungen über die Wärme"; a second part to Professor Carl Neumann's "Elektrische Kräfte"; Vol. II of A. Steinheil and E. Voit's "Handbuch der angewandten Optik"; and Martin, "Die Folgerungen der Bodenreinertragstheorie" (in 3 vols.), Vol. II.

Among the announcements of MM. Gauthier-Villars et Fils of Paris we notice the fourth (and last) volume of G. Darboux's "Leçons sur la théorie générale des surfaces et les applications géométriques du calcul infinitésimal"; a first fascicule (352 pp.) of this volume has appeared. Of Méray's "Leçons nouvelles sur l'analyse infinitésimale et ses applications géométriques" the second volume has been issued, while the third and fourth ("Questions analytiques classiques; applications géométriques") are in preparation. The third and last volume of Fermat's works is in press. Cauchy's "Œuvres complètes" are progressing slowly; vols. 1, 4, 5, 6, 7, 8 of series I, and vols. 6, 7, 8, 9, 10 of series II have appeared so far; vol. 9 of the first series is nearly ready. Eleven of the thirteen volumes of the magnificent edition of Laplace's works are ready; the twelfth is in press. Picard's valuable "Traité d'analyse" is in its third volume, of which two parts (390 pp.) have appeared; the fourth and final volume, on partial differential equations, is in preparation. The second volume of J. Tannery and Molk's "Éléments de la théorie des fonctions elliptiques" is still in press; the whole work will comprise four volumes. Of B. **Niewenglowski**'s "Cours de géométrie analytique," Vol. I was issued in 1894, Vol. II in 1895; the third ("Géométrie dans l'espace, avec une note sur les transformations en géométrie, par E. Borel") is in press. The fourth volume of **Tisserand**'s "Traité de mécanique céleste" has not yet appeared, but is printing; the second volume of **Baillaud**'s "Cours d'astronomie" will soon be out; and a portion of the second volume of P. **Appell**'s "Traité de mécanique" has been issued, while the remaining part and Vol. III are in preparation.

In the number of *Nature* for October 3 is a long list of forthcoming scientific works. Among the works announced are the following: "The year-book of scientific and learned societies of Great Britain and Ireland," comprising lists of the papers read before societies engaged in fourteen departments of research during 1895, to be published by C. Griffin & Co.; "The Indian calendar," containing complete tables for the verification of Hindu and Mohammedan dates for a period of 1600 years (A.D. 300 to 1900) for the whole of India, by **Robert Sewell**, in collaboration with **Sankara Bâlkrishna Dikshit**, with a table of eclipses by Dr. **Schram**, to be published by Swan Sonnenschein & Co; "A practical trigonometry, for the use of engineers, architects, and surveyors," by **Henry Adams**, to be published by Whittaker & Co.; "Miscellaneous papers," by the late Professor H. **Hertz**, translated by D. E. Jones, "Elements of geometry," by **George C. Edwards**, "**Christiansen**'s mathematical physics," translated by W. F. Magie, "Rigid dynamics," by W. J. **Loudon**, "Elements of co-ordinate geometry," by S. L. **Loney**, and "Elementary trigonometry," by **Charles Pendlebury**, to be published by Macmillan & Co. "Dynamics," by P. G. **Tait**, to be published by A. & C. Black; "An introduction to the algebra of quantics," by E. B. **Elliott**, to be published by the Clarendon Press; and, to be published by the Cambridge University Press, "Catalogue of scientific papers, compiled by the Royal Society of London," 1874–1883, Vol. XI; "Scientific papers of **John Couch Adams**," Vol. I, edited by his brother, W. G. Adams, with a memoir by Dr. J. W. L. Glaisher; "Collected mathematical papers of **Arthur Cayley**," Vol. IX; "Treatise on spherical astronomy," by Sir **Robert** S. **Ball**; **Heath**'s "Treatise on geometrical optics," second edition; "Treatise on Abel's theorem," by H. F. **Baker**; "Treatise on the Lunar theory," by E. W. **Brown**; "Elementary treatise on electricity and magnetism," by J. J. **Thomson**; "Treatise on geometrical optics," by R. A. **Herman**; "Euclid," Books XI and XII, by H. M. **Taylor**; and "Mechanics and hydrostatics; part 3, hydrostatics," by R. T. **Glazebrook**.

The announcements of Ginn & Co. include: "Problems in differential calculus," by W. E. **Byerly**; "Molecules and the molecular theory of matter," by A. D. **Risteen**; and "Plane and solid geometry," by W. W. **Beman** and D. E. **Smith**.

NEW PUBLICATIONS.

HIGHER MATHEMATICS.

Abraham ibn Esra. Sefer Ha-Mispar. Das Buch der Zahl. Ein hebräisch-arithmetisches Werk aus dem 12ten Jahrhundert. Uebersetzt und erläutert von M. Silberberg. Frankfurt am Main, 1895. 8vo. 10 and 198 pp. Illustrated. Mk. 4.00

Annuaire de l'École Polytechnique pour l'an 1895. Avec des notes scientifiques, historiques et statistiques et le Mémorial de l'École. Paris, 1895. 8vo. 308 pp. Mk. 2.00

Blank (F.). Ueber die geodätischen Curven auf einem körperlichen Kreisringe. [Progr.] Gera, 1895. 4to. 24 pp. Illustrated. Mk. 1.50

Calinon (A.). Le géométrie à deux dimensions des surfaces à courbure constante. Paris, Gauthier-Villars, 1895. 8vo. Fr. 2.50

Cayley (A.). Collected mathematical papers. Vol. VIII. Cambridge, University Press (New York, Macmillan), 1895. 4to. 4 and 570 pp. 25s.

Chisholm (G.). Algebraisch-gruppentheoretische Untersuchungen zur sphärischen Trigonometrie. [Diss.] Göttingen, 1895. 8vo. 68 pp. 3 plates. Mk. 2.00

Darboux (G.). Leçons sur la théorie générale des surfaces et les applications géométriques du calcul infinitésimal. (In 4 vols.) Vol. IV: Déformation infiniment petite et représentation graphique. Paris, Gauthier-Villars, 1895. 8vo. 1er fasc., pp. 1–352. Whole volume Fr. 15.00

Delassus (E.). Sur les équations linéaires aux dérivées partielles à caractéristiques réelles. [Thèse.] Paris, Gauthier-Villars, 1895. 4to. 67 pp. Fr. 3.50

Grassmann (R.). Die Formenlehre der Mathematik in strenger Formelentwicklung. 4 Teile in 1 Bande, nebst Formelbuch. Stettin, Grassmann, 1895. 8vo. 5 and 12 and 242 pp.; 12 and 189 pp.; 9 and 132 pp.; 3 and 39 pp.; Formelbuch: 24, 27, 14, and 5 pp. Illustrated. Mk. 10.00

Hœfer (F.). Histoire des mathématiques depuis leurs origines jusqu'au commencement du XIXe siècle. 4e édition. Paris, Hachette, 1895. 16mo. 3 and 609 pp. Fr. 4.00

Hurwitz (J.). Ueber eine besondere Art der Kettenbruch-Entwicklung complexer Grössen. [Diss.] Halle, 1895. 8vo. 50 pp. 1 plate.

Israel-Holzwart (K.). Grundlagen und Methoden des tabellarischen Rechnens. Frankfurt am Main, 1895. 4to. 58 pp. Illustrated. Mk. 1.50

Jacobi (C. G. J.). Ueber die vierfach periodischen Functionen zweier Variabeln, auf die sich die Theorie der Abel'schen Transcendenten stützt. (1834.) Herausgegeben von H. Weber. Aus dem Lateinischen übersetzt von A. Witting. Leipzig, Engelmann, 1895. 8vo. 40 pp. Cloth. Mk. 0.70

Jacottet (C.). Ueber die allgemeine Reihenentwicklung der Potentialfunction nach Lamé'schen Producten. [Diss.] Göttingen, 1895. 8vo. 48 pp. Illustrated. Mk. 1.20

Kovalevsky (S.). See **Leffler** (A. C.).

Kovalewsky (S.). Souvenirs d'enfance de Sophie Kovalewsky, écrits par elle-même, et suivis de sa biographie par M^{me} A. Ch. Leffler, duchesse de Cajanello. Paris, Hachette, 1895. 16mo. 10 and 334 pp. Fr. 3.50

Kronecker (L.). Werke, herausgegeben auf Veranlassung der Königl. Preussischen Akademie der Wissenschaften von K. Hensel. (In 4 vols.) Vol. I. Leipzig, Teubner, 1895. 4to. 10 and 484 pp. 1 portrait. Mk. 28.00

Klein (F.). Ueber die Beziehungen der neueren Mathematik zu den Anwendungen. Antrittsrede, gehalten am 25. Oktober 1880 bei Uebernahme der damals an der Universität Leipzig neuerrichteten Professur für Geometrie. (Besonderer Abdruck aus Band XXVI der *Zeitschrift für mathematisch-naturwissenschaftlichen Unterricht.*) Leipzig, Teubner, 1895. 8vo. 12 pp.

Laplace. Œuvres complètes, publiées sous les auspices de l'Académie des sciences par les secrétaires perpétuels, avec le concours de MM. Puiseux, F. Tisserand, J. Hoüel et Souillard. (In 13 vols.) Vol. XI: Mémoires extraits des recueils de l'Académie des sciences. Paris, Gauthier-Villars, 1895. 4to. Fr. 20.00

Leffler (A. C.). Sonya Kovalevsky: a biography. By Anna Carlotta Leffler, duchess of Cajanello, and sisters Rajevsky: being an account of her life, by Sonya Kovalevsky. Translated by A. de Furuhjelm and A. M. Clive Bayley, with a biographical note by Lily Wolffsohn. London, Unwin, 1895. 8vo. 380 pp. 6s.

——— Sonia Kovalevsky: biography and autobiography. I: Memoir by by A. C. Leffler (Edgren), duchessa di Cajanello. II: Reminiscences of childhood, written by herself. Translated into English by Louise von Cossel. London, Scott (New York, Macmillan), 1895. 12mo. 316 pp. Frontispiece. $1.25

Ljungh (A. T.). Ueber isoptische und orthoptische Kurven. [Diss.] Lund, 1895. 8vo. 50 pp. 1 plate.

Loney (S. L.). The elements of co-ordinate geometry. London and New York, Macmillan, 1895. 12mo. 10 and 420 pp. Cloth. $1.25

Manaira (A.). Sopra una certa rappresentazione piana dell' elissoide di rivoluzione e sulla applicazione di essa ai calcoli geodetici. Padova, Prosperini, 1895. 8vo. 56 pp. 1 plate.

Mathematical Questions and Solutions. From the *Educational Times*, etc. Edited by W. J. C. Miller. Vol. 63. London, Hodgson, 1895. 8vo. 128 pp. 6s. 6d.

Nielsen (N.). Om en Klasse bestemte Integraler og nogle derved definerede semiperiodiske Funktioner. Kjöbenhavn, 1895. 4to. 80 pp. Mk. 4.00

Petersen (J.). Forelæsninger over Funktionsteori. Hæfte II og III. Kjöbenhavn, 1895. 8vo. 112 and 144 pp. Mk. 7.60

Plücker (J.). Gesammelte wissenschaftliche Abhandlungen. Im Auftrag der Königl. Gesellschaft der Wissenschaften zu Göttingen herausgegeben von A. Schoenflies und F. Pockels. Vol. I: Mathematische Abhandlungen, herausgegeben von A. Schoenflies. Leipzig, Teubner, 1895. 8vo. 36 and 620 pp. 1 portrait. Mk. 20.00

Rebstein (J.). Bestimmung aller reellen Minimalflächen, die eine Schaar ebener Kurven enthalten, denen auf der Gauss'schen Kugel die Meridiane entsprechen. Berlin, 1895. 8vo. 30 pp. Mk. 1.20

Riemann (B.). Sur le nombre des nombres premiers inférieurs à une grandeur donnée. Traduit par L. Laugel. Paris, Gauthier-Villars, 1895. 8vo. Fr. 0.75

Rosenhain (G.). Abhandlungen über die Functionen zweier Variabler mit vier Perioden, welche die Inversen sind der ultra-elliptischen Integrale erster Klasse. (1851.) Herausgegeben von H. Weber. Aus dem Französischen übersetzt von A. Witting. Leipzig, Engelmann, 1895. 8vo. 94 pp. Cloth. Mk. 1.50

Schlömilch (O.). Compendium der höheren Analysis. Band II: Vorlesungen über einzelne Theile der höheren Analysis. 4te Auflage. Braunschweig, Vieweg, 1895. 8vo. 10 and 546 pp. Illustrated. Mk. 9.00

Sievert (H.). Ueber Thetafunktionen, deren Charakteristiken aus Fünfteln ganzer Zahlen bestehen. II. [Progr.] Bayreuth, 1895. 8vo. 36 pp.

Silberberg (M.). See **Abraham ibn Esra.**

Snyder (V.). Ueber die linearen Complexe der Lie'schen Kugelgeometrie. [Diss.] Göttingen, 1895. 8vo. 48 pp. Illustrated. Mk. 1.40

Weber (E. von). Die singulären Lösungen der partiellen Differentialgleichungen mit drei Variablen. [Habilitationsschrift.] München, 1894. 8vo. 30 pp.

Weierstrass (K.) Mathematische Werke. Herausgegeben unter Mitwirkung einer von der Königl. Preussischen Akademie der Wissenchaften eingesetzten Commission. Vol. II: Abhandlungen II. Berlin, Mayer & Müller, 1895. 4to. Mk. 21.00

Weld (L. G.). A short course in the theory of determinants. 12mo. 238 pp. $1.90

Zeuthen (H. G.). Anvendelse af Differentialregning paa algebraiske Ligninger. Kjöbenhavn, 1895. 4to. 24 pp. [Lithogr.] Mk. 0.80

ELEMENTARY MATHEMATICS.

Aleu (M. L.) Elementos de matemáticas. Geometria. Valdemoro, 1895. 4to. 12 and 206 pp. Illustrated. Mk. 6.50

——— Elementos de matemáticas. Trigonometria rectilinea. Madrid, 1895. 4to. 52 pp. Illustrated. Mk. 2.50

Andriani (A.). Elementi di algebra ad uso dei licei e degli istituti tecnici secondo gli ultimi programmi con numerosa raccolta di esercizi. 2a edizione migliorata. Napoli, Trani, 1895. 16mo. 390 pp. Fr. 4.00

Bardey (E.). Arithmetische Aufgaben, nebst Lehrbuch der Arithmetik. 9te Auflage. Leipzig, Teubner, 1895. 8vo. 12 and 270 pp. Cloth. Mk. 2.40

——— Methodisch geordnete Aufgabensammlung, mehr als 8000 Aufgaben enthaltend über alle Theile der Elementar-Arithmetik. 21ste Auflage. Leipzig, Teubner, 1895. 8vo. 14 and 330 pp. Cloth. Mk. 3.20

Bork (H.). Periodische Decimalbrüche. [Progr.] Berlin, 1895. 4to. 42 pp.

Breuer (P. J.). Das Notwendigste über die natürlichen Logarithmen. Leipzig, Teubner, 1895. 8vo. 40 pp. Mk. 0.80

Brooksmith (E. J.). See **Woolwich Mathematical Papers.**

Capelli (A.). Lezioni di algebra complementare ad uso degli aspiranti alla licenza universitaria in scienze fisiche e matematiche. Napoli, Pellerano, 1895. 8vo. 12 and 526 pp. Illustrated. Fr. 8.00

Ciamberlini (C.). Elementi di geometria, ad uso delle scuole tecniche, normali e professionali. Parte I: Planimetria. Montegiorgio, Delbello, 1895. 16mo. 149 pp.

Cominotto (E.). Trisezione approssimata dell' angolo. Padova, Tip. del Seminario, 1895. 8vo. 7 pp.

——— Sopra una disposizione particolare dei triangoli simili: seguito di una nota che verrà pubblicata nel periodico *Lugli* di Roma. Padova, Tip. del Seminario, 1895. 8vo. 7 pp.

Frankenbach (W.). Die Harmonikalen der Mittelpunkte der Berührungskreise eines Dreiecks in Bezug auf dasselbe. Liegnitz, 1895. 8vo. 32 pp. Illustrated. Mk. 1.50.

Gauss (F. G.). Fünfstellige vollständige logarithmische und trigonometrische Tafeln. 46ste Auflage. Halle, 1895. 8vo. 166 and 35 pp. Cloth. Mk. 2.50

Gérard (L.). Géométrie à l'usage des candidats aux baccalauréats de l'enseignement secondaire. Paris, 1895. 8vo. Fr. 3.00

Gremigni (M.). La teorica dei poligoni equivalenti. Firenze, Paggi, 1895. 8vo. 28 pp. Fr. 1.00

Holzmüller (G.). Methodisches Lehrbuch der Elementar-Mathematik. Im engsten Anschluss an die neuen Lehrpläne. (In 3 Teilen.) Teil I. 2te Auflage. Leipzig, Teubner, 1895. 8vo. 8 and 230 pp. Cloth. Mk. 2.40

Laisant (C. A.) et **Lemoine** (E.). Traité d'aritmétique; suivi de notes sur l'ortografie simplifiée, par P. Malvezin. Paris, Gauthier-Villars, 1895. 16mo. 8 and 174 pp.

Laisant (C. A.). Recueil de problèmes de mathématiques, classés par divisions scientifiques, contenant les énoncés avec renvoi aux solutions de tous les problèmes posés, depuis l'origine, dans divers journaux: *Nouvelles annales de mathématiques*, *Journal de mathématiques élémentaires et de mathématiques spéciales*, *Nouvelle correspondance mathématique*, *Mathésis*. (In 7 vols.) Vol. III: Algèbre; théorie des nombres; probabilités; géométrie de situation. Paris, Gauthier-Villars, 1895. 8vo. Fr. 6.00

Legendre. Elementi di geometria, con giunte e modificazioni di A. Blanchet. Versione italiana, conforme ai programmi ministeriali, autorizzata ed approvata da A. Blanchet, con note ed aggiunte di G. Hueber. 26ma edizione. Napoli, Morano, 1894. 16mo. 310 pp. Illustrated. Fr. 4.00

Lemoine (E.). See **Laisant** (C. A.) et **Lemoine** (E.).

Lesky (A.). Bestimmung der Maxima und Minima quadratischer Gleichungen. Klagenfurt, 1895. 8vo. 20 pp. Mk. 1.00

Livini (A.). Elementi di geometria e topografia. Vol I: Geometria piana e stereometria. 3a edizione. Napoli, Pontieri, 1894. 16mo. 140 pp. Illustrated. Fr. 2.00

Lomholt (A.). Regulære Polyedre. En elementær Skitse. Viborg, 1895. 8vo. 30 pp. 2 plates. Mk. 0.70

Lucas (É.). L'arithmétique amusante. Amusements scientifiques pour l'enseignement et la pratique du calcul. (Introduction aux *Récréations mathématiques.*) Paris, Gauthier-Villars, 1895. 8vo. Fr. 7.50

Malagodi (A.). Nozioni d'algebra elementare, con numerosi esercizi e problemi, ad uso della terza classe delle scuole tecniche. 2a edizione riveduta e corretta. Mirandola, Cagarelli, 1895. 8vo. 64 pp. Fr. 0.50

Marti (R.). Operazioni del calcolo e loro analogia. Lecce, Tip. cooperativa, 1895. 8vo. 36 pp.

Monnet (L. F.). Cours élémentaire d'algèbre et de trigonométrie théorique et pratique, à l'usage des lycées, des collèges, etc. 11e édition. Paris, André, 1894. 18mo. 324 pp. Illustrated.

——— Cours élémentaire de géométrie théorique et pratique, à l'usage des lycées, des collèges, etc., contenant un très grand nombre d'exercices résolus et à résoudre. 15e édition. Paris, André, 1895. 18mo. 360 pp.

Monti (P.). Regola generale per la soluzione grafica della trisezione dell' angolo, colla dimostrazione di G. Schiaparelli. Milano, Bombelli, 1895. 4to. 7 pp. 1 plate.

Moser (J.). Harmonische Strahlen und Punkte. Breslau, 1895. 4to. 12 pp. Illustrated. Mk. 1.20

Ortu Carboni (S.). Geometria descrittiva elementare ed alcune sue applicazioni (proiezioni ortogonali). Vol. II. Torino, Paravia, 1895. 8vo. 4 and 158 pp. Illustrated. Fr. 3.00

Otto (F. A.). Die rechnende Geometrie (System Otto), eine unerschöpfliche Fundgrube für mathematische Forschung. Mit Anhang: Das geometrische Problem des 20sten Jahrhunderts (Otto's Problem). Tegel, 1895. 12mo. 36 pp. Illustrated. Mk. 1.00

——— Ein Problem der Rechenkunst. Rationelle Methode zur Auflösung von numerischen Gleichungen mit einer Unbekannten (beliebiger Grad und jede Form). 2te Auflage. Tegel, 1895. 12mo. 32 pp. Mk. 0.50

Persiani (O.). Elementi di geometria secondo Euclide, ad uso dei licei. Libri V–VI. Roma, Spellani, 1895. 8vo. 52 pp. Illustrated.

Pesci (G.). Trattato elementare di trigonometria piana e sferica; con appendice. Livorno, 1895. 8vo. 330 and 80 pp. Illustrated. Fr. 5.00

Pietzker (F.). See **Unterrichtsblätter.**

Russell (J. W.). An elementary treatise on pure geometry. With numerous examples. 12mo. 324 pp. $2.60

Schwalbe (B.). See **Unterrichtsblätter.**

Stromillo (S.). Lezioni elementari di geometria per le scuole secondarie. Parti I e II. Napoli, Muca, 1894. 16mo. 107 and 135 pp. Illustrated. Fr. 4.25

Thompson (W. T.). Scholarship algebra. London, Hughes, 1895. 8vo. 280 pp. 2s. 6d.

Unterrichtsblätter für Mathematik und Naturwissenschaften. Herausgegeben von B. Schwalbe und F. Pietzker. Braunschweig, Vieweg, 1895. 4to. 5 numbers per year. Mk. 3.00

Vine (G. T.). Mathematical note-book, specially prepared for the use of pupils, teachers, scholarship and certificate candidates. Newport, 1895. 8vo. 88 pp. Cloth. $0.50

Weeks (W.). Exercises in Euclid, graduated and systematized. 18mo. 80 pp. $0.60

Woolwich Mathematical Papers for admission into the Royal Military Academy, for the years 1885–94. Edited by E. J. Brooksmith. London and New York, Macmillan, 1895. 12mo. Cloth. $1.50

APPLIED MATHEMATICS.

Albrich (K.). Der Unterricht in Mechanik auf geschichtlicher Grundlage. Hermannstadt, 1894. 4to. 35 pp. Illustrated. Mk. 1.50

Alheilig et **Roche** (C.). Traité des machines à vapeur, rédigé conformément au programme du cours de machines à vapeur de l'École centrale. (In 2 vols.) Vol. I. Paris, Gauthier-Villars, 1895. 8vo. 10 and 604 pp. Illustrated. Fr. 20.00

Assmann (R.). See **Fortschritte der Physik.**

Bertoldo (G.). Compendio di termodinamica applicata. Vol. I. Torino, Unione tipografico-editrice, 1895. 8vo. 239 pp. Illustrated. Fr. 4.00

Börnstein (R.). See **Fortschritte der Physik.**

Bourlet (C.). Traité des bicycles et des bicyclettes, suivi d'une application à la construction des vélodromes. (Encyclopédie scientifique des aide-mémoire.) Paris, Gauthier-Villars, 1895. 8vo. Cloth. Fr. 3.00

Brown (H. T.). 500 meccanismi, scelti fra i più importanti e recenti, riferentisi alla dinamica, idraulica, idrostatica, pneumatica, macchine à vapore, molini, torchi, orologeria ed altre macchine. 2a edizione, compilata sulla 16a edizione inglese da F. Cerruti. Milano, 1895. 12mo. 182 pp. Illustrated. Fr. 3.00

Cart (A.) et **Portes** (L.). Calcul des ponts métalliques à poutres droites, à une ou plusieurs travées, par la méthode des lignes d'influence. Formules et tables. Paris, Baudry, 1895. 8vo. 8 and 292 pp.

Cazes (L.). Stéréoscopie de précision: théorie et pratique. Paris, 1895. 8vo. 72 pp. Illustrated. Fr. 2.50

Cerruti (F.). See **Brown** (H. T.).

Clerke (A. M.). The Herschels and modern astronomy. (Century Science Series.) London and New York, Macmillan, 1895. 12mo. 6 and 224 pp. Portraits. Cloth. $1.25

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Articles for insertion should be addressed to the

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41 East 49th Street, New York City.

The following dates have been fixed for the meetings of the Society:

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Printed by J. S. Cushing & Co., Norwood, Mass.

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OF THE

AMERICAN MATHEMATICAL SOCIETY.

CONTINUATION OF THE BULLETIN OF THE NEW YORK MATHEMATICAL SOCIETY.

A HISTORICAL AND CRITICAL REVIEW
OF MATHEMATICAL SCIENCE.

EDITED BY
THOMAS S. FISKE, ALEXANDER ZIWET, FRANK MORLEY.

2ND SERIES, VOL. II. NO. 5.

FEBRUARY, 1896.

NEW YORK:
PUBLISHED FOR THE SOCIETY
BY MACMILLAN & CO.
1896.

Entered at the Post-office at New York, N. Y., as second-class matter.
PUBLISHED MONTHLY, EXCEPT AUGUST AND SEPTEMBER.

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REMARKS ON THE PROGRESS OF CELESTIAL MECHANICS SINCE THE MIDDLE OF THE CENTURY.

PRESIDENTIAL ADDRESS DELIVERED BEFORE THE AMERICAN MATHEMATICAL SOCIETY, DECEMBER 27, 1895.

BY DR. G. W. HILL.

THE application of mathematics to the solution of the problems presented by the motion of the heavenly bodies has had a larger degree of success than the same application in the case of the other departments of physics. This is probably due to two causes. The principal objects to be treated in the former case are visible every clear night, consequently the questions connected with them received earlier attention; while, in the latter case, the phenomena to be discussed must ofttimes be produced by artificial means in the laboratory; and the discovery of certain classes of them, as, for instance, the property of magnetism, may justly be attributed to accident. A second cause is undoubtedly to be found in the fact that the application of quantitative reasoning to what is usually denominated as physics generally leads to a more difficult department of mathematics than in the case of the motion of the heavenly bodies. In the latter we have but one independent variable, the time; while in the former generally several are present, which makes the difference of having to integrate ordinary differential equations or those which are partial. Thus it happens that, while the science of astro-mechanics is started by Newton, that of thermal conductivity receives its first treatment, at the hands of Fourier, more than a century later. In addition to these two causes, ever since the discovery of the telescope the application of optical means to the discovery of whatever might be found in the heavens has always had a fascination for mankind. And, as the ability to co-ordinate and correlate the facts observed much enhances the enjoyment of scientific occupation, it has resulted that many who began as observers ended as mathematical astronomers. Thus our science has had relatively a large number of cultivators.

A thoroughly satisfactory history of our subject is yet to be written. We have only either slight sketches of the whole, or elaborate treatments of special divisions of the science, and none of them coming down to recent times. Among the former may be mentioned Gautier's *Essai historique sur le pro-*

blème des trois corps, which appeared in 1817. Also Laplace's historical chapters in the last volume of the *Mécanique Céleste*. Todhunter's History of the theories of attraction and the figure of the earth is an example of the latter class. Such books as Todhunter's — of which Delambre has given an earlier example in his *Histoire de l'Astronomie* — can hardly be regarded as history; they resemble rather extensive tables of contents of the literature examined, accompanied by short comments. However, in many cases, they are more useful to the student than formal histories would be, as, when judiciously compiled, they may, as epitomes in our libraries, take the place of a large mass of scientific literature. The History of Physical Astronomy, by Robert Grant, is a book that comes down to 1850, and professedly covers the whole of our subject. But only one third of this book is devoted to astro-mechanics, the rest dealing with what is really observational and descriptive astronomy. Moreover, the author indulges so much in diffusive veins of writing, that but a small fraction of the 200 pages is really given to purely historic statement. As far as the Lunar Theory is concerned, the third volume of M. Tisserand's *Traité de Mécanique Céleste* constitutes a fair history. But it must be borne in mind that the author's plan is to notice only the disquisitions having a first-class importance; hence his history is incomplete in this respect.

In America we are not well situated for investigations of this character, on account of the meagreness of our libraries. Of no inconsiderable number of memoirs and even books, having at least some importance in our subject, there exist no copies in the United States. Hence, should an American be inclined to undertake the task of writing the history of our subject, he must at least perform some of the work abroad.

In the present discourse it is proposed to touch very lightly the more important steps made since the middle of the century, the time at our disposal not admitting fuller treatment.

And first we will take up Delaunay's method, proposed for employment in the lunar theory, but quite readily extended to all classes of problems in dynamics. The first sketch of this method, given of course by the author himself, appeared in the *Comptes Rendus* of the Paris Academy of Sciences, in 1846. It professes to be merely an extract from a memoir offered for publication in the collections of the Academy, which must, however, have been afterwards withdrawn to make place for the two volumes of the *Théorie du Mouvement de la Lune*. When this extract is compared with the earlier chapters of the latter work, it is perceived that Delaunay has, to some extent, modified and improved his method in the interim between 1846 and 1860. In this long period nothing appeared from the author on this subject. He must have been profoundly engaged in applying his method to the motion of the moon. Tisserand's exposition of this method is somewhat more brief than the author's own. But when the necessary modifications are introduced into Delaunay's procedures, to make them applicable to the more general case of the motion of a system of bodies, the establishment of the formulas can be rendered still more brief.

There is one point in reference to Delaunay's method which, as far as I am aware, has escaped notice. This method consists in a series of operations or transformations, in each of which the position of the moon in space is defined by six variables, the number three being doubled in order that the velocities, as well as the co-ordinates, may be expressed without differentials. The aim of the transformations is to make one half of these, which Poincaré has called the linear variables, continually approach constancy, while the other half, named the angular variables, continually approach a linear function of the time. But at any stage of the process the position of the moon, as well as its velocity, is definitely fixed by the six variables produced by the last transformation, provided that the proper degree of variability is attributed to them, just as, before any transformation was made, the six elements of elliptic motion, usually denominated osculating, defined them; the point of difference to be noticed being that the more the transformations are multiplied, the more complex becomes the character of the expression of the former quantities in terms of the latter. But, however great may be the number of transformations, the series evolved have always one consistent trait, viz., that the angular variables are involved in them only through cosines or sines of linear functions of these variables, the linear functions being formed with integral coefficients. Now, as in all this work we are obliged to employ infinite series, the question of their convergence is an extremely important one. The inquiry in this respect may be divided into two parts, mainly independent of each other. These are, convergence as respects the angular variables, and convergence as respects the linear variables. The first part is much the more simple. Regarding each of the coefficients of the series we employ as a whole, that is, representing it by a definite integral, it is quite easily perceived that the said series are both legitimate and convergent when, giving the angular variables the utmost range of values, still no two of the bodies can occupy the same point of space. In the contrary case the series are evidently divergent. This condition affords certain limiting conditions for the values of the linear variables. Could we trace these limiting conditions through all the transformations, and obtain by comparison the formulas to which these tend when the number of transformations is made infinite, we should be in possession of the conditions of stability of motion of the system of bodies. The second part of the inquiry relates to the expression of the mentioned coefficients by infinite series proceeding according to powers and products of certain parameters which are functions of the linear variables. It is well known that, in the case of elliptic elements, Laplace and Cauchy almost simultaneously showed that the series are convergent when the eccentricity does not exceed a fraction which is about two-thirds. The determination of the conditions of convergence, after certain transformations have been made in the signification of the elements, is undoubtedly a more complex problem; nevertheless, it seems to be within the competency of analysis as it exists at present.

The discovery of the criterion for the convergence of series proceeding according to powers and products of parameters is due to Cauchy, and is a most remarkable contribution to the science of mathematics. Supposing that the parameters begin from zero values, this criterion amounts to saying that the moment the function, which the series is to represent, ceases to be holomorphic, or becomes infinite, that moment the series ceases to be convergent. Consequently, if a space, having as many dimensions as there are parameters in the case, be conceived, and a surface be constructed in it formed by the consensus of all the points where the considered function ceases to be holomorphic, then, provided the values of the parameters define a point within this surface, that is, on the same side where lies the origin, the series will be convergent. Generally this surface will be closed, and, within it, the function will not take infinity as its value.

Without any mathematical reasoning the propriety of the principle just enunciated may be perceived. Since it is possible for the series in powers and products to give only one value for the function, the moment the latter may have any one of several values, the series fails to give them all; and, as there is no reason why any particular value should be selected, the conclusion must be that it does not represent any of them. Also, it is easy to see that, when the function takes infinity as its value, the series fails to represent it.

In applying this principle to the series involved in the treatment of the problem of many bodies by Delaunay's method, it appears, at first sight, as if we must have some finite representation of the coefficients in question in order to discover the particular points at which they cease to be holomorphic, such, for instance, as is given by an algebraic or transcendental equation. But this is not imperative, as it is often possible to make this discovery from certain recognized properties of the function considered, without being in possession of its form explicitly or implicitly. It appears probable that, in the class of cases considered, the mentioned coefficients can be represented by multiple definite integrals, all taken between the limits 0 and π, the independent variables being those which have been denominated angular. Such functions are always holomorphic, provided that the expressions under the signs of integration are themselves holomorphic between the mentioned limits. If the statement just made be admitted, although it may be impossible to write explicitly the mentioned expressions, we may, nevertheless, be certain that they remain holomorphic, provided that the linear variables, which may be the same as the parameters considered, are so restricted in their range of values that no matter what values the angular variables receive, no distance between any two bodies of the system can vanish. Or, in other words, that the R of Delaunay must never become infinite. Thus it seems probable that the conditions of convergence for Delaunay's series are precisely identical with those for the stability of motion of the system.

The series arising in Delaunay's method, as applied to the moon, contain five parameters; the number would be six were the moon's mass not neglected. We should also have six in the application of the method to two planets moving about the sun; however, should we employ the well-known functions $b_s^{(i)}$ of Laplace, the number would be reduced to five. It ought to be possible, therefore, after the performance of a limited number of operations, to assign limiting values to these parameters, below which the series would certainly be convergent. This also involves the possibility of finding limits to the errors committed by truncating the series at a certain order of terms. Again, provided the time is limited to a certain interval, the capacity of these truncated series for representing the co-ordinates of the planets could be shown by giving superior limits to the errors necessarily involved.

One more remark may be made before we leave Delaunay's method. In every operation or transformation half the integrals are obtained without the intervention of the time, and from these solely are obtained the ranges of values for all the linear variables. As no integrating divisors appear in their expressions, it follows that the question of stability is not affected in any way by the vanishing of these. Moreover, the presence of a libration in the angle of operation does not necessitate any change in the procedure. The integrating divisors which appear in the expressions for the angular variables, obtained through quadratures, may cause difficulty, but this can generally be removed by a modification of the parameters employed in the development of the coefficients in series. Beyond this it does not seem necessary to attend particularly to the terms which Professor Gyldén has designated as critical.

To give a succinct idea of the scope of this method, it may be said that it is applicable whenever, in the system, the planets maintain their order of succession from the sun. In systems where that undergoes change, as is the case with the group of minor planets, supposing their action on each other is sensible, it is not applicable.

Delaunay's method has not yet received all the developments and applications it is susceptible of.

The treatise of Hansen on the shortest and most ready method of deriving the perturbations of the small planets was published in the interval 1857–1861. But as the principles on which it is founded had been elaborated and communicated to the public some years earlier, it is, perhaps, more properly to be assigned to the first half of the century. In consequence, I pass it over with this slight mention.

Perhaps the most conspicuous labors in our subject, during the period of time we consider, are those of Professor Gyldén and M. Poincaré. We will limit our attention, for the remainder of this discourse, to the consideration of these investigations.

Professor Gyldén began work with the methods of Hansen and was gradually led to modifications of them looking towards their use for indefinite lengths of time. This quality has latterly become imperative with him, and he has recently published the first volume of what is evidently intended to be a lengthy work entitled *Traité Analytique des Orbites Absolues des Huit Planètes Principales.* To show the drift of Professor Gyldén's investigations, we cannot do better than give an analysis of this volume. At the outset the author introduces a class of curves he names periphlegmatic, that is, curves which surround a flame. The definition of this sort of curve is that it describes continually the space between two concentric spheres, and, at every point, turns its concavity towards the intersection of the radius vector with the inner sphere. In an application to the solar system, the sun is supposed to occupy the common centre of the spheres. The investigation is at first limited to the case where this curve is plane. A differential equation of the second order is derived which the radius vector of this curve satisfies, the independent variable being the angle described. The perpendicular distance between the spheres is called the diastem. The spheres are supposed to be drawn so that they touch the curve at the points where the radius becomes a maximum or minimum. Thus, in some cases, the spheres are regarded as fixed, in others, as movable. In the latter case, however, the sum of their radii is supposed to remain constant. Thence we have two groups of periphlegmatic curves; those with constant and those with variable diastems. The author gives examples of both these groups, in most cases of which the line of apsides is variable, and considers the situation and density of the points of intersection of these curves with themselves.

The idea of an absolute orbit of a planetary body is this: an oval symmetrical with regard to an axis movable in space. While the axis remains constant in length (the half of it is called the protometre), the velocity of its motion may vary, and the diastem may also vary. Professor Gyldén, however, admits into the expressions of these variations only terms whose period would become infinite did the planetary masses vanish. These terms he calls elementary. But elementary terms in the diastem and the longitude of the perihelion can produce terms in the co-ordinates having periods which differ but little from the time of revolution of the planet. These are also called elementary terms. But the two classes are distinguished, the first as being of the type (A), and the second as of the type (B). In all the formulas relative to this matter the author insists on keeping the arc described by the radius as the independent variable.

The co-ordinates are only approximately given by the preceding apparatus of expressions. They must then have certain complements added to them; these, however, are all composed of terms which would vanish with the planetary masses.

In deriving the elementary terms in the radius of a planet through the integration of a linear differential equation of the second order, Professor Gyldén attaches much price to his method of establishing the convergence of the series formed by the successive terms. As the latter are obtained through division by divisors of the order of the planetary masses, it might be feared that some of them would turn out to be very large. But the author prevents this by retaining in the coefficient of the dependent variable in the differential equation a quantity equivalent to the sum of the squares of all the coefficients in the integral. This is named the horistic or limiting function. It is plain such an expression could be introduced in the mentioned coefficient, provided that the linear equation is the truncated form of an equation containing the cube of the variable. And in the problem of planetary motion the approximations may always be so ordered that this shall be the case.

With regard to the co-ordinate which exhibits the departure of the planet from a fixed plane, Professor Gyldén does not greatly deviate from the procedure of Hansen in following the displacement of the instantaneous plane of the orbit. Only here, as in the preceding treatment of the radius, he would sharply distinguish the elementary and non-elementary terms.

At this point is introduced certain new nomenclature. As before we had diastem now we have anastem to denote the product of the radius and the sine of the inclination; and what has generally been called the true argument of the latitude is here called the anastematic argument. Any angular magnitudes which are constantly moving through the circumference are astronomic arguments; and when they have the same mean velocity of rotation they are isokinetic; and isokinetic arguments are homorhythmic when, in each revolution through the circumference, they always retake together the same corresponding points. In like manner, the true anomaly is the diastematic argument, and we have diastematic and anastematic coefficients and moduli. It will be seen from this that Professor Gyldén does not shrink from imposing on us the labor of learning new terms.

Thus far we have been engaged in deriving the equations of the path followed by a heavenly body; it remains to show how we may find the point on that path occupied by the body at a given moment. There is then necessary an equation between the time and the variable assumed as independent, that is, the orbit longitude, or, more properly, the amount of angle described by the radius vector. If we suppose the absolute orbit to be described by the planet so that equal areas are passed over by the radius in equal times, it is plain that, on the attainment of a given longitude, a definite amount of time must have elapsed since the epoch. This is what Professor Gyldén calls the *reduced* time; and he computes the difference between it and the actual time required by the theory of gravity for the planet to arrive at the stated direction. This mode of proceeding does not differ from Hansen's except in the point that the absolute orbit is substituted for a fixed ellipse.

But this gives us correctly only the orbit longitude; for the radius and the latitude, which correspond in the absolute orbit to this reduced time, are not quite those which the planet has at the actual time. Consequently, Professor Gyldén proposes to compute two corrections, the one to be applied to the product of the eccentricity into the cosine of the true anomaly, the other to the sine of the latitude. Also the reduction of the orbit longitude to the plane of reference must be manipulated so that it comes out correctly.

The employment of the orbit longitude as independent variable throughout all the integrations necessitates a mass of very intricate transformations of terms from one shape into another. Also the integrations which bear on elementary terms must be kept distinct from those which bear on non-elementary terms. A degree of complexity is thus imparted to the subject, which makes it difficult to see when one has really gathered up all the warp and woof of it. Professor Gyldén has nowhere removed the scaffolding from the front of his building and allowed us to see what architectural beauty it may possess; it is necessary to compare a large number of equations scattered through the volume before one can opine how the author means to proceed.

The advantages claimed for the method are that it prevents the time from appearing outside the trigonometrical functions, and that it escapes all criticism on the score of convergence. The first is readily conceded, but many simpler methods possessing this advantage are already elaborated, and it is not so clear that the second ought to be granted.

No completely worked out example of the application of this method has yet been published. The great labor involved will naturally deter investigators from employing it.

In 1890 was published the memoir of M. H. Poincaré, entitled *Sur le problème des trois corps et les équations de la dynamique*, and which obtained the prize of the King of Sweden. Most of the results of this memoir were worked over and presented anew with greater elaboration and clearness by their author in *Les Méthodes Nouvelles de la Mécanique Céleste.* Here we find a large number of new and very interesting theorems.

First is to be noted the class of particular solutions in the problem of the motion of a system of material points which are now named *periodic solutions.* The initial relative positions and velocities of the several points are so adjusted that, after the lapse of a definite time, the latter retake them. Hence is evident a method which may be employed to elaborate this special case of motion, viz., by the tentative process with mechanical quadratures. M. Poincaré has divided this sort of solutions into three classes, of which, however, the second and third are not essentially different. He has shown that, in the latter classes, the values of the arbitrary constants of the problem must be so adjusted that no secular inequalities, or, as Professor Gyldén calls them, elementary terms, may arise. The number and variety of these particular solutions is far greater than one would at first sight imagine.

We come now to a second class of particular solutions named by the author *asymptotic.* It arises from the consideration of solutions differing very little from periodic solutions. Here we have to deal with linear differential equations having periodic coefficients. The integrals of these contain in their terms exponential factors, and on the nature of the exponents of these factors depends the quality of the resulting solutions. M. Poincaré has named these exponents *characteristic.* They are roots of an algebraic equation of a degree equal to the number of dependent variables involved in the question. If any of these roots are imaginary with real portions or wholly real, we are in presence of asymptotic solutions. The algebraic equation mentioned contains the unknown only in even powers; hence the characteristic exponents are in pairs having the same absolute value, but with contrary signs. In all the cases presented by astronomy, where, on account of the near approach to circular motion, a periodic solution can be taken as a first approximation, it appears that the squares of the characteristic exponents are all real and negative. Thus, there is no call here to consider this sort of solution, and this fact must much diminish the interest of the astronomer in it. M. Poincaré has, however, elaborated it with great pains, showing how the effect of higher powers of the deviations from the periodic solution may be taken into account. The series resulting are, nevertheless, divergent, as in other cases.

The second volume of the *Méthodes Nouvelles* is devoted to the elaboration and consideration of various processes for developing the integrals of planetary motion according to the powers of a small parameter. The chief of these are due to Professor Newcomb and MM. Lindstedt and Bohlin; but M. Poincaré has augmented the number of them by introducing modifications of his own. All involve the principle of recurrence; that is, the first step is the only one which is independent, the following depend on all that precede. These methods, in their general aspect, do not differ from the old developments in powers of the disturbing fórce, except the operations are so adjusted that the time never escapes from the trigonometric functions. This is accomplished by greatly augmenting the number of the elementary arguments, and by supposing that the rate of motion of each of these is developable according to integral powers of the before-mentioned parameter, or, in some cases, of its square root.

When there is more than one elementary argument, the series obtained in all these ways are pronounced to be generally divergent in the rigorous sense of the word. M. Poincaré brings forward several methods of proof of this. The first depends on the presence of small divisors in the expressions of the coefficients. However, when we do not insist on developments in powers of a parameter, this method of proof has no application. Another method is derived from the principle that two characteristic exponents vanish for every uniform integral that exists. But the integrals which necessitate this conclusion must not only be uniform, they must be valid for every possible case of the problem. Now the integrals known as those of the conservation of living forces and of areas are of this nature; but the integrals derivable from the series of Delaunay, Newcomb, and Lindstedt are valid only for a limited range in the values of the linear variables. For instance, in the problem of the three bodies, if the deformation of the triangle formed by these bodies is such that we cannot find any two sides, one of which sustains to the other an invariable relation of greater to less, we cannot apply the mentioned series. And here it is well to note that the defect of convergence does not arise from the application of the processes of integration, but already exists in the development of the perturbative function before integration commences. Thus Delaunay's development of this function at the beginning of his lunar theory is divergent and illusory, unless we have the lunar radius in apogee always less than the solar radius in perigee, and that without regard to the mode of expressing the coefficients. Some of the particular integrals, relied upon by M. Poincaré to establish the vanishing of all the characteristic exponents in case we accept M. Lindstedt's series as valid, lie, so to speak, on the boundary of the domain in which these series are convergent.

In the third place an appeal is made to the alleged non-existence of analytic and uniform integrals beyond those already known. Were this non-existence clearly established it would decide the question on the side where M. Poincaré has placed himself. But, at least as far as the non-existence of integrals of this nature in a limited domain for the linear variables is concerned, the proof given for it is quite defective. This proof consists in ascertaining how these integrals, supposing them to exist, would behave, should we attempt to derive periodic solutions from them. It is difficult to present this matter without the assistance of algebraic formulæ; nevertheless, it may be attempted. Let there be a number of equations whose left members are formed by the product of two factors. When we pass to a periodic solution, one of these factors becomes zero. What conclusion can we draw from each of the thus modified equations? Evidently one of two things: either the remaining factor of the left member is infinite and the right member indeterminate, or it is finite and the right member a vanishing quantity. Now in case we are obliged to accept the first conclusion, were it only but once, M. Poincaré has demonstrated the non-existence of integrals; but, granting that it is proper in every case to accept the latter conclusion, the demonstration fails. Now he declines to consider the latter alternative, saying that he does not believe that any problem of dynamics, presenting itself naturally, occurs where the right members of the mentioned equations would all vanish. But it should be borne in mind that, while they do not vanish in the general equations, the adjustment of the values of the linear parameters required by the passage to a periodic solution may bring about their vanishing. Thus, in the lunar theory, a periodic solution is brought about by making $e = 0$, $e' = 0$, and $\gamma = 0$, the result is the vanishing of every coefficient having any of these quantities as a factor.

M. Poincaré appeals in another place to the fact that the Lindstedt series, if convergent, would establish the non-existence of asymptotic solutions. But this observation is irrelevant for the reason that the domains of the two things are quite distinct.

In any case where Lindstedt series are applicable there are no asymptotic solutions, and, where there are asymptotic solutions, Lindstedt's series would be illusory.

We owe much to M. Poincaré for having commenced the attack on this class of questions. But the mist which overhangs them is not altogether dispelled; there is room for further investigation.

KRONECKER'S LINEAR RELATION AMONG MINORS OF A SYMMETRIC DETERMINANT.

BY PROFESSOR HENRY S. WHITE.

AMONG the minors of any determinant there exist well-known identical relations; those of lowest order, the quadratic relations, being readily obtained by the expansion of a determinant in which at least one pair of rows or of columns are identical. If, however, the original determinant is symmetrical, there are identities of a lower order than the quadratic, the linear identities first formally noticed by Kronecker in 1882.* These linear relations, published with no hint as to the manner of their discovery, are suggestive of a certain formula in such constant use as to have become a commonplace in the transformations of the Theory of Invariants of linear substitutions. The latter formula, however, relates to products of two determinant-factors, while Kronecker's is linear; but the latter uses double indices for the constituents, and herein lies the resemblance. By virtue of the ordinary process of multiplication of two determinants, Kronecker's theorem is easily proved to be a consequence from the other identity. Both are equally general, hence it seems likely that the earlier may have been the source of the later. This theory I will develop inductively, using for the sake of brevity determinants of three rows, and obtaining a typical linear relation among three-rowed minors of a six-rowed symmetric determinant.

Form an array of three rows of six constituents each:

$$\begin{matrix} a_1 & a_2 & a_3 & a_4 & a_5 & a_6 \\ b_1 & b_2 & b_3 & b_4 & b_5 & b_6 \\ c_1 & c_2 & c_3 & c_4 & c_5 & c_6 \end{matrix}$$

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* *Sitzungsberichte der Berliner Akademie*, 1882, p. 824. See also C. Runge: Die linearen Relationen zwischen den verschiedenen Subdeterminanten symmetrischer Systeme. *Jour. für r. u. a. Math.*, vol. 93 (1882), pp. 319–327.

Denote the determinant of any three columns in this array by the three indices in proper order, as

$$\begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} = (1\,2\,3).$$

The identity employed in transforming invariants will then be written as follows:

$$(1)\quad (1\,2\,3)(4\,5\,6) \equiv (1\,2\,4)(3\,5\,6) + (1\,2\,5)(4\,3\,6) + (1\,2\,6)(4\,5\,3).$$

The correctness of this is seen upon developing in minors of the first three, and of the last three rows the vanishing determinant:

$$\begin{vmatrix} a_1 & a_2 & a_3 & a_4 & a_5 & a_6 \\ b_1 & b_2 & b_3 & b_4 & b_5 & b_6 \\ c_1 & c_2 & c_3 & c_4 & c_5 & c_6 \\ 0 & 0 & a_3 & a_4 & a_5 & a_6 \\ 0 & 0 & b_3 & b_4 & b_5 & b_6 \\ 0 & 0 & c_3 & c_4 & c_5 & c_6 \end{vmatrix} \equiv 0.$$

Multiplying the determinants of each pair in (1), column by column, we have in each product-determinant constituents all of the same form, which can be abbreviated thus:

$$a_1a_4 + b_1b_4 + c_1c_4 = 1,\, 4,$$

or

$$a_ia_k + b_ib_k + c_ic_k = i,\, k.$$

Identity (1) will thus become:

$$(2)\quad \begin{vmatrix} 1,4 & 2,4 & 3,4 \\ 1,5 & 2,5 & 3,5 \\ 1,6 & 2,6 & 3,6 \end{vmatrix} \equiv \begin{vmatrix} 1,3 & 2,3 & 4,3 \\ 1,5 & 2,5 & 4,5 \\ 1,6 & 2,6 & 4,6 \end{vmatrix} + \begin{vmatrix} 1,4 & 2,4 & 5,4 \\ 1,3 & 2,3 & 5,3 \\ 1,6 & 2,6 & 5,6 \end{vmatrix}$$

$$+ \begin{vmatrix} 1,4 & 2,4 & 6,4 \\ 1,5 & 2,5 & 6,5 \\ 1,3 & 2,3 & 6,3 \end{vmatrix}$$

We may observe, from the definition of the symbol, that $i, k = k, i$. Accordingly if the 36 quantities i, k be arranged as a six-rowed determinant, in the order denoted by their double indices, that determinant will be symmetric; that is, if

$$A_{ik} = A_{ki} = (a_ia_k + b_ib_k + c_ic_k),$$

then, (3)

$$D_6 = |A_{ik}|(i,\, k = 1, 2, 3, 4, 5, 6)$$

will be a symmetric determinant, of which the determinants in (2) above will be third minors.

The identity (2) *is a Kronecker relation among three-rowed minors of the six-rowed symmetric determinant* D_6.

Such relations are thus derived for certain symmetric six-rowed determinants. D_6, however, is of a highly specialized type; it is the discriminant of the sum of three 6-ary squares. Is the relation (2), established for this special type, valid for all symmetric determinants of six rows? It is; for it involves no constituent from the principal diagonal, so that the 18 parameters of our 6×3 array are available for representing the 15 constituents $\left(\frac{6 \cdot 5}{1 \cdot 2} = 15\right)$, lying outside the principal diagonal of any given symmetric 6-rowed determinant.

The same consideration can be relied upon in adapting this proof to any Kronecker relation among m-rowed minors of a $2m$-rowed symmetric determinant, since always

$$\frac{2m(2m-1)}{2} < 2m^2.$$

For convenience of reference, I subjoin the general form of this relation as enunciated by Kronecker, *loc. cit.*:

$$|a_{gh}| = \sum_r |a_{ik}|,$$

$$\left(\begin{array}{ll} g=1, 2, \cdots, m; & h=m+1,\ m+2, \cdots, 2m \\ i=1, 2, \cdots, m-1, r; & k=m+1,\ m+2, \cdots, r-1,\ m,\ r+1 \cdots 2m \end{array}\right).$$

NORTHWESTERN UNIVERSITY,
EVANSTON, ILLINOIS, *December* 10, 1895.

ON THE LISTS OF ALL THE SUBSTITUTION GROUPS THAT CAN BE FORMED WITH A GIVEN NUMBER OF ELEMENTS.

BY DR. G. A. MILLER.

T. P. KIRKMAN published in 1863 the first extensive list of all the transitive substitution groups that can be formed with a given number of letters. A number of interesting facts are associated with this list. Before entering upon its discussion we shall give a brief account of the more important direct steps towards the formation of such lists.

Paolo Ruffini published a work* in 1799 in which we do not only find groups divided into transitive,* intransitive, primitive and non-primitive, but also an attempt to give all the orders of the groups that can be formed with five elements. This work is especially interesting because it contains some concepts which have been attributed to Cauchy for a long period of years.†

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* The complete title of this work in two volumes is, "*Teoria generale delle equazioni, in cui si dimostra impossibile la soluzione algebraica delle equazioni generali di grado superiore al quarto, di Paolo Ruffini,*" Bologna, 1799.

In 1845 Augustin Cauchy published in *Comptes Rendus*, vol. 21, pp. 1363–1369, lists of all the possible orders of the groups that can be formed with two, three, four, five, and six letters. He determines the orders of the intransitive, non-primitive and primitive groups separately. His lists contain a few orders for which no group of the given class exists, and the list of the orders of the non-primitive groups of degree six does not contain 36. There are no other omissions. Some theorems relating to group construction precede these lists.

In 1850 J. A. Serret published in *Liouville's Journal* complete lists of all the substitution groups that can be formed with four and with five letters. No special notation is employed, but the groups are written in ordinary language in the summaries on pages 52, 53 and 65–67. The facts that these lists are complete and that they contain the groups themselves instead of their orders only are evidences of a significant step forward even if the author confined himself to a very small number of letters.

Emile Mathieu continued the lists which Cauchy began, by enumerating all the possible orders of groups which contain seven and eight letters. These lists are given in *Comptes Rendus*, vol. 46, pp. 1048 and 1208 respectively.

Kirkman's List.

This is found in "*Proceedings of the Literary and Philosophical Society of Manchester*" (1863), vol. 3, pp. 133–152.‡ A table of corrections containing ten additional groups and some changes of the notation of the groups in the list is given by the same author on p. 172, vol. 4, of the same journal.

We reproduce below, the first part of this list, through degree

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[*] Cf. JORDAN, *Traité des substitutions*, preface VII.; MATHIEU, *Comptes Rendus*, 46, pp. 1048 and 1049.

[†] Cf. H. BURKHARDT, *Schlömilch's Zeitschrift*, 1892, supplement, p. 159. In the same article is found a discussion of the part of Ruffini's work which relates to substitution groups, pp. 133–137. We refer to this article for a more complete account of Ruffini's work.

[‡] The heading of the article in which this list is given is, "The complete theory of groups, being the solution of the mathematical prize question of the French Academy for 1860. By the Rev. Thos. P. Kirkman, M.A., F.R.S., and honorary member of the literary and philosophical societies of Manchester and Liverpool."

four, for the purpose of adding clearness to the following remarks. It may be observed that the group of two letters is not given in the list.

$3\cdot 1=1+2_{3}$, $Q=1$; $3\cdot 2=1+2_{3}+3_{21}$, $Q=1$,
$4\cdot 1=1+2_{4}+1_{2^2}$, $Q=3$; $4\cdot 2=1+2_{4}+3_{2^2}+2_{21^2}$, $Q=3$,
$4\cdot 3=1+8_{31}+3_{2^2}$, $Q=1$; $4\cdot 6=1+8_{31}+3_{2^2}+6_{4}+6_{21^2}$, $Q=1$.*

Explanations of the notation. The first members of these equations give the orders of the groups. The second members give the enumeration of the different types of substitutions found in the groups. The subscripts indicate the degree of the cycle or cycles that compose the substitutions of the given type. Q stands for the number of conjugates of the group in the corresponding symmetric group.

The list and the table of corrections contain all the transitive groups through degree seven with exception of the group of degree two mentioned above. The following six groups of degree eight are omitted.

ORDER.	NUMBER.	GROUP.
16	1	$\left\{ \begin{matrix} (ae\,.\,bf\,.\,cg\,.\,dh)B'(abcd\,.\,efgh) \text{ or} \\ (aebf\,.\,cdgh) \text{ cyc } (ac\,.\,bd)(ef\,.\,gh), \end{matrix} \right\}$
24	1	$(abcd\,.\,efgh)$ pos $(ae\,.\,bg\,.\,cf\,.\,dh)$,
96	1	· $(ae\,.\,bf\,.\,cg\,.\,dh)\{(abcd)_{12}(efgh)_{12}\}$ ter,
	2	$(ae\,.\,bg\,.\,cf\,.\,dh)\{(abcd)_{12}(efgh)_{12}\}$ ter,
192	1	$\{(abcd)$ all $(efgh)$ all$\}$ sex $(ae\,.\,bf\,.\,cg\,.\,dh)$,
576	1	$\{(abcd)$ all $(efgh)$ all$\}$ pos $(aebf\,.\,cg\,.\,dh)$.

The two given groups of order 16 are conformal † and they have the same number of conjugates in the symmetric group of degree eight. It is, therefore, impossible to determine which of these is omitted, since Kirkman's notation for the two is exactly the same. The fact that his notation involves such uncertainty is an objection to it. A more serious objection is furnished by the fact that the notation does not enable one to write directly the substitutions of the group.

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* The list does not contain the four-group. This is given in the table of corrections together with two missing groups of degree six, viz., $(abcdef)_{12_2}$ and $\pm(abcdef)_{24}$. The list is complete for the degrees three, five, and seven.

† For brevity we use the term "conformal groups" for non-conjugate groups, all of whose substitutions are of the same type. Kirkman does not give any conformal groups in the list itself. The table of corrections, however, contains a group which is conformal to one in the list, viz., a G_{32}. Cf. KIRKMAN, *Mathematics from Educational Times*, vol. 38, pp. 80, 81.

It appears quite likely that Kirkman regarded the two given groups of order 16 as conjugate without going through the full proof. Two other omissions may be accounted for in almost the same way. Kirkman's group of order 96, and the first group of this order given above, as well as the two positive transitive groups of order 192, are also conformal. In each of these cases the omitted group has 210 conjugates in the symmetric group of degree eight, while the given group has only 105, so that it is easy to determine which of the conformal groups are in the list.

The other three omitted groups cannot be accounted for in the same way, since Kirkman's notation for these groups would be quite distinct from that used for other groups in the list.

It may be observed that while Kirkman omitted six transitive groups of degree eight there are twenty-four such omissions in Cayley's list,* which was published about thirty years later.

The six groups, which are not given in Kirkman's list and table of corrections, have been published as follows:

Three, those of orders 16 and 96, in Cayley's list mentioned above. The other three were first published in this journal; those of orders 192 and 576,† May, 1893, and the one of order 24, April, 1894.

For degree nine Kirkman again omitted six groups, viz.:

ORDER.	NUMBER.	GROUP.
54	1	(*abc*.*def*) cyc (*abc*.*ghi*) cyc (*adg*.*bfi*.*ceh*) × (*ad*.*be*.*cf*.*gh*),
	2	{ [{(*abc*)all(*def*)all}pos(*ghi*)all] sex(*adg*.*bfi*.*ceh*) or (*abc*.*def*)cyc(*abc*.*ghi*)cyc(*adg*.*bfi*.*ceh*)cyc × (*ad*.*bf*.*ce*), }
72	1	(*abcd*.*efgh*) cyc (*ac*.*ef*.*gh*)(*aci*.*bgh*.*dfe*.) cyc,
108	1	[{(*abc*) all (*def*) all} pos (*ghi*) all] sex × (*adg*.*bfi*.*ceh*) cyc (*ad*.*be*.*cf*.*gh*),
162	1	{(*abc*) pos (*def*) pos (*ghi*) pos + (*abc*) neg (*def*) neg (*ghi*) neg} (*adg*.*beh*.*cfi*) cyc,
648	1	{(*abc*) all (*def*) all (*ghi*) all} pos × (*adg*.*beh*.*cfi*) cyc (*ad*.*be*.*cf*).

It is impossible to decide which one of the two groups 54_2 is given by Kirkman, for they are conformal, and one has 1120 while the other has 3360 conjugates in the symmetric group of degree nine. Kirkman gives 1680 as the value of Q, which is certainly erroneous.

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* *Quarterly Journal of Mathematics*, vol. 25, pp. 137–155.

† These two are given in Professor COLE's "Note on the substitution groups of six, seven, and eight letters." This "note" is also found in *Quarterly Journal of Mathematics*, vol. 26, pp. 372–376.

It may be well to state here that numerous errors occur in Kirkman's notation for other groups, especially in the higher orders. He never gives any more groups of a given order than actually exist. It therefore seems quite likely that the errors in notation are due to insufficient study of the groups under consideration. This view is, perhaps, supported by the author's statement in the introduction to the list, viz., "If any (transitive groups whose degree is less than 11) have escaped me, it is the fault, not of my method, but of my carelessness." For degree ten the list is so inaccurate as to be of little value.

In 1872, Camille Jordan gave an enumeration of the numbers of primitive groups for each degree through degree seventeen in *Comptes Rendus*, 75, p. 1757. Only two omissions have been published, one of degree[*] nine and the other of degree twelve.[†]

About twenty years later[‡] Askwith, who apparently was unacquainted with Serret's list,[§] published two articles in the *Quarterly Journal of Mathematics*[||] in which he endeavored to give all the groups for each degree from three to nine, together with the methods by which they may be found. He referred only to Serret's work,[¶] although two more comprehensive treatises[**] had been published.

In the following number of the same journal, Professor Cayley gave the results which Askwith had obtained, together with some additions, in a condensed and perspicuous form.[††] In the introduction to this list he explains his notation and calls attention to some very important principles in group construction. The list is very inaccurate. Twenty intransitive and twenty-eight transitive groups were omitted.

Since Cayley's notation is far superior to Kirkman's, and since the corrections to Cayley's list are published in the same notation as the list itself, it may be of especial interest to give an outline of these corrections.

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[*] This journal, vol. 3, p. 245.

[†] *Ibid.* vol. 1, 2d series, p. 255.

[‡] The most important work in the line of our subject, which was published in this period, is that of VERONESE, *Annali di Mathematica* (2), vol. 11, pp. 93–233. This contains lists of the groups of degrees 2, 3, 4, 5, and 6, on pp. 153, 157, 158, 167, 168, 177–190, 232, 233. Both the transitive and the intransitive groups are enumerated.

[§] Cf. *Quarterly Journal of Mathematics*, vol. 24, p. 111.

[||] Vol. 24, pp. 111–167 and 263–331.

[¶] *Ibid.* pp. 111, 144, 270.

[**] JORDAN, *Traité des Substitutions*, 1870, and NETTO, *Substitutionentheorie*, 1882. CAUCHY's work, *Exercices d'analyse*, vol. 3, pp. 151–252 (1844), is not quite as comprehensive as Serret's.

[††] *Quarterly Journal of Mathematics*, vol. 25, pp. 71–88 and 137–155.

In the "Note on the substitution groups of six, seven, and eight letters,"* Professor Cole gave all the missing groups except two. Some errors in this note are corrected by its author in the *Quarterly Journal*.† It remains to add that in the enumeration at the end of the note, 199 and 48 should be replaced respectively by 200 and 50. The two missing groups were published separately in this journal.‡

It is very probable that the groups through degree eight have been completely determined. The following enumeration may therefore be accepted with a great deal of confidence as a permanent one.

Number of letters	2	3	4	5	6	7	8
Number of groups	1	2	7	8	37	40	200
Transitive groups	1	2	5	5	16	7	50

Through degree five Serret's list of 1850 was complete, as stated before.

The work begun by Askwith in the two articles mentioned above is continued in vol. 26 of the same journal. His attempt to determine all the transitive groups of degree nine was not wholly successful.§

All the groups of degree nine ‖ and the transitives of degrees ten ¶ and eleven have recently been published by Professor Cole, and the intransitives of degree ten ** by the author. From the preceding it will be seen that these enumerations according to degree have been pursued to the following extent. All the possible groups through degree ten, the transitives through degree eleven,†† and the primitives through degree seventeen have been determined.

It should not be inferred that the enumerations according to degree are the only ones which have received attention. In addition to these, there are enumerations of the primitive groups according to class,‡‡ of simple groups according to order,[*] of groups whose orders are composed of given kinds of factors,[†] of regular groups according to degree,[‡] etc.

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* This journal, May, 1893; *Quarterly Journal of Mathematics*, vol. 26, pp. 372–376.

† Vol. 27, p. 50.

‡ Vol. 3, pp. 168 and 242–244. Professor Moore informed me that this group had been determined in practically the same manner by NOETHER, *Mathematische Annalen*, vol. 15, pp. 90–95.

§ Cf. this journal, vol. 2, p. 250.

‖ *Quarterly Journal of Mathematics*, vol. 26, pp. 376–388. Cf. *Ibid.* vol. 27, p. 102, foot-note.

¶ *Ibid.* vol. 27, pp. 39–50. Cf. this journal, vol. 1, 2d series, pp. 67–72.

** *Quarterly Journal of Mathematics*, vol. 27, pp. 99–118.

†† The transitives of degree twelve will appear in vol. 28, *Quarterly Journal*.

‡‡ JORDAN, *Comptes Rendus*, vol. 75, p. 1757; NETTO, *Theory of Substitutions* (COLE's edition), pp. 133–138.

Methods of Group Construction.

Most of the lists which have been mentioned are preceded by brief explanations of the methods employed by the authors, and references are generally given to works where more complete explanations may be found. Among the separate articles on this subject, special attention should be called to the two following: "On the construction of intransitive groups"[§] by Professor Bolza, and "Construction of composite groups"[‖] by Professor Hölder.

The methods of finding all the groups of a given degree may be divided into two great classes: (1) those by means of which we prove the existence of groups and find their distinguishing features, and (2) those by means of which we prove the non-existence of given types of groups in certain regions. The method of the second class employed in the following examples has extensive application in the construction of such non-primitive groups.

In constructing the non-primitive groups of degree ten, it is necessary to inquire whether there is a non-primitive group with five systems of two letters which is simply isomorphic to $(abcde)$ pos. If we interchange the systems only according to the substitutions of the given group in five letters and preserve the same order of the letters of the systems throughout, we evidently obtain $(abcde\,.\,fghij)$ pos. The average number of letters in this group is eight,[¶] and the average number in the required group is nine.[¶] If such a group exists, it must therefore be possible to solve the Diophantine equation,

$$20a + 15b = 60 \ (a = 0, 2, 4; \ b = 0, 2).$$

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[*] Hölder, *Mathematische Annalen*, vol. 40, pp. 55–89; Cole, *American Journal*, vol. 14, pp. 378–388, and vol. 15, pp. 303–316; Frobenius, *Berliner Mittheilungen*, 1893, pp. 221–229; *Sitzungsberichte*, 337–345.

[†] Netto, *Theory of Substitutions* (Cole's edition), pp. 145–149; Young, *American Journal*, vol. 15, pp. 124–179; Cole and Glover, *ibid.* pp. 191–221; Hölder, *Annalen*, vol. 43, pp. 301–413, and *Göttinger Nachrichten*, 1895, pp. 211–229; Sylow, *Acta Mathematica*, vol. 11, pp. 201–256; Burnside, *Proceedings of the London Mathematical Society*, vol. 25, pp. 9–18.

[‡] Kempe, *Philosophical Transactions*, vol. 177, pp. 37–43; Cayley, *American Journal*, vol. 11, pp. 139–157; Levavasseur, *Comptes Rendus*, vol. 120, pp. 822–825 and 899–903; vol. 121, pp. 238–240.

[§] *American Journal of Mathematics*, vol. 11, pp. 195–214.

[‖] *Mathematische Annalen*, "Bildung zusammengesetzter Gruppen," pp. 321–422. A series of articles on elementary methods of group construction are now being published in the *American Mathematical Monthly*.

[¶] Frobenius, *Crelle*, vol. 101, p. 287. Cf. "A simple proof of an elementary theorem in substitutions," etc. This journal, current vol., p. 75.

This equation depends directly upon the fact that the substitutions of the non-primitive group must interchange all the systems represented by the letters of the corresponding substitution of (*abcde*) pos, and leave the remaining systems unchanged. Since the given Diophantine equation has clearly no solution, no non-primitive group of the required type exists.

Similarly there can be no non-primitive group of degree twelve whose six systems of two letters correspond to the elements of (*abcdef*) pos and whose base is unity unless the following equation can be solved,

$$40a + 45b + 144c = 360$$

$$(a = 0, 2, 4, 6;\ b = 0, 2, 4;\ c = 0, 2).$$

This has evidently no solution. It is not necessary to examine the corresponding symmetric groups; for, if it is possible that a non-primitive group in the given number of systems is simply isomorphic to either of them, its self-conjugate subgroup corresponding to the alternating group must be intransitive. This could, therefore, be regarded as the base, and its systems of intransitivity as the systems of the required non-primitive group.

It remains to notice a certain dual interpretation of the elements of a non-regular primitive group. If we represent the subgroups which do not contain $a, b, \cdots$ by $A, B, \cdots$, then will every substitution of the group $(abc.de\cdots)$ interchange these subgroups according to an identical substitution in the capital letters $(ABC.DE\cdots)$. We may, therefore, think of the substitutions as interchanging the elements or as interchanging these subgroups. This dual interpretation is not always so interesting in the other groups, since there need not then be a 1, 1 correspondence between the elements and the capital letters.*

Whenever all the capital letters stand for subgroups whose degree is one less than the degree of the group, such a 1, 1 correspondence must exist even if the group is intransitive or non-primitive.

LEIPZIG, *October*, 1895.

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* Cf. MOORE, this journal, vol. 1, 2d series, pp. 61, 62.

ON CAUCHY'S THEOREM CONCERNING COMPLEX INTEGRALS.

BY PROFESSOR MAXIME BÔCHER.

THE following arrangement of the proof of Cauchy's theorem, that the integral of a holomorphic function in a simply connected region is a function of the limits of integration, but not of the path, is more elementary than the proofs ordinarily met with. Like the proofs given by Goursat* and Jordan,† it avoids the use of double integrals or the calculus of variations, and it has the advantage over these proofs not merely of brevity, but also of avoiding more or less complicated considerations of limits. The materials from which the following proof is built up are so familiar to all students of the theory of functions,‡ and the consequent probability that this method of putting them together is not unknown, is so great, that I should hardly have ventured to publish it, had it not seemed to possess peculiar pedagogical advantages. These depend, apart from its elementary character, on the fact that almost all the steps involved establish theorems, or illustrate methods, with which the student must become familiar sooner or later.

The theory of integrals of a function of a complex variable depends, as is well known, on the theory of integrals of the form

$$I = \int_{(a, b)}^{(x, y)} (Pdx + Qdy),$$

in which P and Q are real functions of the two real variables x, y. We will begin with the theorem

S being any region in the xy-plane in which P and Q are single-valued and have continuous first partial derivatives, a necessary and sufficient condition that the integral I should depend merely on the limits of integration is the existence in S of a single-valued function $\phi(x, y)$, for which $\frac{\partial \phi}{\partial x} = P$, $\frac{\partial \phi}{\partial y} = Q$.

For, if such a ϕ exists, $Pdx + Qdy$ is the complete differential of ϕ, and the integral I is the limit of the sum of the increments of ϕ, which we get in going along the path of integration from (a, b) to (x, y). But the sum of these increments is $\phi(x, y) - \phi(a, b)$, so that I is independent of the path of integration. Conversely, if I is independent of the path, it will be itself (the lower limit (a, b) being regarded as constant) a function $\phi(x, y)$ of the sort we want; for, in the first place, I is now single-valued in S by hypothesis. Its increment when x alone changes is

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* See HARKNESS and MORLEY'S *Theory of Functions*, p. 164.

† *Cours d'Analyse* (2d ed.), vol. 1, p. 185.

‡ See, for instance, PICARD, *Traité d'Analyse*, vol. 1, pp. 81–83, and BOUSSINESQ, *Cours d'Analyse infinitésimal*, vol. 2, pp. 6–12.

$$\Delta_x I = \int_{x,\,y)}^{(x+\Delta x,\,y)} (P dx + Q dy).$$

In this integral the path of integration may be taken as a straight line, so that y is constant. We get then

$$\Delta_x I = \int_x^{x+\Delta x} P(x, y) dx = \Delta x \cdot P(x + \theta\Delta x, y),$$

where $0 \leqq \theta \leqq 1$. Dividing through by Δx, and taking the limit as Δx approaches zero, we get $\frac{\partial I}{\partial x} = P(x, y)$. Similarly, we find $\frac{\partial I}{\partial y} = Q(x, y)$.

Obviously, then, a necessary condition that I should be independent of the path is that at every point of S

$$(A) \qquad \frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x},$$

since both sides of this equation will be $\frac{\partial^2 \phi}{\partial x \partial y}$. It remains to show that this is also a sufficient condition, provided that the region included between the paths of integration belongs wholly to the region S, as would necessarily be the case if S were simply connected. We first prove the following:

Lemma.—*Equation* (A) *is a sufficient as well as a necessary condition for rectangular regions* S *whose sides are parallel to the axes of* x *and* y.

To establish this lemma, we must show that if equation (A) holds throughout our rectangle, a single-valued function ϕ exists there, such that $\frac{\partial \phi}{\partial x} = P$, $\frac{\partial \phi}{\partial y} = Q$. Such a ϕ we get here again in the integral I itself, provided that, in order to ensure that I shall be a single-valued function of x, y, we agree to make the path of integration consist of two straight lines,—from (a, b) to (x, b) and from (x, b) to (x, y). Owing to the restricted shape of S, these two lines will always lie wholly in S. We have then

$$\phi = \int_a^x P(x, b) dx + \int_b^y Q(x, y) dy,$$

$$\frac{\partial \phi}{\partial y} = Q(x, y),$$

$$\frac{\partial \phi}{\partial x} = P(x, b) + \int_b^y \frac{\partial Q(x, y)}{\partial x} dy = P(x, b) + \int_b^y \frac{\partial P(x, y)}{\partial y} dy = P(x, y).$$

From the lemma just proved, it follows at once that equation (A) is the necessary and sufficient condition that the integral I should vanish when taken around any closed path which lies within or upon the boundary of a rectangle of the sort we have just considered.

We will now state the final proposition we have to prove in the following form:

Whatever shape the region S may have, the sufficient as well as the necessary condition that the integral I taken in the positive direction around the complete boundary of any region S' lying actually within[*] *S should be zero, is equation* (A).

To prove this, let us divide up that portion of S where S' lies into rectangles by lines parallel to the axes of x and y, and take these rectangles so small[†] that those rectangles which lie partly within and partly without S' shall lie wholly within S. Then S' will be divided into a number of pieces, some of which are rectangles, while the others are parts of rectangles. The integral taken in the positive direction around the complete boundary of S' is obviously equal to the sum of the integrals taken in the positive direction around the boundary of each of the small pieces into which we have divided S'. But each of these last integrals vanishes if equation (A) holds throughout S, since each of the paths lies within or on the boundary of a rectangle of the sort considered in the preceding lemma, throughout which equation (A) holds. The integral taken around the complete boundary of S' will therefore vanish, if (A) holds throughout S.

Bearing in mind that

$$\int (u+vi)(dx+idy)=\int (udx-vdy)+i\int (vdx+udy),$$

we infer at once from the proposition just proved the following proposition:

u and v being single valued real functions of the real variables x, y in a region S of the xy-plane, and having there continuous first partial derivatives, the necessary and sufficient condition that $\int (u+vi)(dx+idy)$ taken around the boundary of a region S' which lies within S should be zero is:

$$\frac{\partial u}{\partial x}=\frac{\partial v}{\partial y};\quad \frac{\partial u}{\partial y}=-\frac{\partial v}{\partial x}.$$

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[*] That is, the boundary of S' must not coincide with that of S at any point.

[†] It should be carefully noticed that these rectangles are not "infinitesimal," *i.e.* that we are not going to allow them to decrease indefinitely.

It might be objected that in the above proof we assume that the region S' around whose boundary we integrate lies within another region S, throughout which the conditions of continuity and the equation (A) are satisfied. In the first place, however, this will be true in all cases to which we ordinarily apply the theorem;[*] and in the second place the proof can easily be so modified as to obviate this difficulty, at least for all ordinary shapes of the boundary. All we should have to do would be by a slight extension of the method to prove the lemma given above, not for rectangles with sides parallel to the axes of x and y, but for regions bounded by three sides of such a rectangle, and on the fourth side by a curve which is cut by no line parallel to the two parallel straight sides in more than one point. It will clearly be possible, in any ordinary case, to take the rectangles into which we cut up the region S' so small that the pieces of rectangles which occur near the boundary of S' shall be of this nature.

HARVARD UNIVERSITY, *December*, 1895.

NOTES.

A REGULAR meeting of the AMERICAN MATHEMATICAL SOCIETY was held in New York, Saturday afternoon, January 25, at three o'clock, the President, Dr. HILL, in the chair. There were seventeen members present. One nomination for membership was received. The report of the auditing committee, appointed at the preceding meeting to examine the Treasurer's accounts, was presented and accepted. The following papers were read:

(1) Professor HENRY S. WHITE: "Kronecker's linear relation among the minors of a symmetric determinant."

(2) Professor H. TABER: "On certain sub-groups of the general projective group."

In the absence of Professor White his paper was read by Mr. Ling.

THE following mathematical courses are offered in the University of Leipzig for the summer semester of the present year: — Professor Scheibner: Theory of numbers; — Professor Neumann: Selected chapters in mathematical physics; — Professor Lie: Theory of groups (continuous transformation groups); — Professor Mayer: Differential equations of dynamics; — Professor Engel: Ordinary differential equations; — Dr. Scheffers: (1) Theory of surfaces; (2) Introduction to higher mathematics; — Dr. Hausdorff: (1) Analytic geometry; (2) Introduction to the mathematical theory of assurance. Owing to the illness of Professor Bruns, the astronomical courses have not yet been announced.

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[*] The only case when it would not be true would be when the path of integration meets a natural boundary of the function we are integrating.

The number of mathematical students at Leipzig (not including those studying mathematics as a minor) is at present 42; last summer it was 37, the preceding winter 25.

THE Royal Academy of Science and Letters of Denmark offers a gold medal to the author of a memoir containing a notable contribution to the solution of the following problem: *To indicate the necessary and sufficient criterions to decide whether an algebraic equation with given numerical coefficients belongs, or does not belong, to the class of Abelian equations.*

In presenting the question, it is stated that notwithstanding the great progress of algebra in modern times, with the exception of equations soluble by square roots, the problem to determine whether or not a given algebraic equation of unknown origin admits of solution, is to-day substantially where it was at the beginning of the century. However, it can hardly be doubted that the evolution of algebra during this epoch should render some advancement possible and furnish, in some cases at least, new methods of determining whether such an equation is of the class soluble through the extraction of roots.

Competitors should send their manuscripts before the end of October, 1896, to the Secretary of the Academy, Professor G. H. Zeuthen, of the University of Copenhagen. Decision will be rendered during the month of February following, after which authors may withdraw their memoirs.

REPORT OF THE TREASURER FOR THE YEAR 1895.

The report which I have the honor to submit to you exhibits the receipts and disbursements for the year ending December 27, 1895.

In the receipts are included: 1st, the amounts received for dues and initiation fees for the year 1895; 2d, the amounts received for dues of previous years; 3d, the amounts received for dues of the year 1896; 4th, the amounts received from subscriptions to and sales of the Bulletin; and 5th, the amounts received from subscriptions to fund for publication of papers presented at the Mathematical Congress of the Columbian Exposition.

The total receipts of the Society during the past year, as shown by the following itemized statement, amount to $1423.26. The disbursements for the same period amount to $862.63. The itemized vouchers for these disbursements are herewith submitted in Exhibit A.

The Society's present cash assets, including in these a total of $192.50 of subscriptions to the publication fund mentioned above, on deposit in the Fifth Avenue Bank of New York, are $1677.30. The net increase in assets during the past year is $560.63.

R. S. WOODWARD, *Treasurer.*

NEW YORK, *December* 27, 1895.

The Treasurer in Account with the American Mathematical Society.

Dr.	
1895.	
Dec. 27.	
To balance on hand Dec. 27, 1894	$1116.67
To cash received for dues and initiation fees for the year ending Dec. 27, 1895	1156.26
To cash received for dues for the year 1893	10.00
To cash received for dues for the year 1894	85.00
To cash received for dues for the year 1896	20.00
To cash received from subscriptions to and sales of the Bulletin	119.50
To cash received from subscriptions to fund for publication of papers of Mathematical Congress of Columbian Exposition	32.50
	$2539.93

	Cr.
1895.	
Dec. 27.	
By cash paid Macmillan & Co. for publication of 8 numbers of Bulletin and extra copies thereof	$556.50
By cash paid William R. Jenkins for stamped envelopes; for printing notices of meetings, circulars, ballots, and blanks; and for stationery	168.56
By cash paid for expenses of publication committee, including clerical assistance, postage, expressage, services of messengers, and telegrams	113.57
By cash paid as contribution of American Mathematical Society to the New York Alliance of Scientific Societies	24.00
By balance on deposit in the Fifth Avenue Bank of New York, N. Y.	1677.30
	$2539.93

REPORT OF AUDITING COMMITTEE.

The Committee finds that all sums entered as receipts in the Treasurer's books are properly accounted for by the balance remaining to the Society's credit in the bank and by proper vouchers for expenditures.

F. N. COLE,
J. B. CHITTENDEN,
POMEROY LADUE.

NEW YORK, *January* 25, 1896.

NEW PUBLICATIONS.

HIGHER MATHEMATICS.

Cantor (M.). Vorlesungen über Geschichte der Mathematik. (In 3 Bänden.) Band III: Vom Jahre 1668 bis zum Jahre 1759. (In 3 Abteilungen.) Abteilung II: Die Zeit von 1700 bis 1726. Leipzig, Teubner, 1896. 8vo. pp. 253–472. Mk. 6.00

Diophantus Alexandrinus. Opera omnia cum græcis commentariis. Edidit et latine interpretatus est P. Tannery. Vol. II: Pseudepigraphica; testimonia veterum; scholia vetera, etc. Lipsiæ, Teubner, 1895. 8vo. 48 and 298 pp. Mk. 5.00

D'Ovidio (E.). Geometria analitica. Torino, Bocca, 1896. 8vo. 16 and 443 pp. Fr. 10.00

Favaro (A.). Lezioni di geometria proiettiva. 3a edizione, riveduta ed aumentata. Padova, Sacchetti, 1895. 8vo. 8 and 260 pp. Fr. 5.00

Guyot (E.). Simple exposé des relations à établir entre les équations et leurs racines. Paris, May et Motteroz, 1895. 8vo. 85 pp.

Hermite (C.). Sur les nombres de Bernouilli. Bruxelles, 1895. 8vo. 10 pp. Fr. 1.00

Kovalewsky (S.). Souvenirs d'enfance, écrits par elle-même et suivis de sa biographie par A. C. Leffler. Nouvelle édition. Paris, Hachette, 1895. 16mo. 10 and 304 pp. Fr. 3.50

Las de Bosredon (V.). Considérations sur l'intersection des coniques. Bruxelles, 1895. 8vo. 14 pp. Fr. 1.00

Leffler (A. C.). See **Kovalewsky** (S.).

Müller (W.). Transformationstheorie der Monge-Ampère'schen partiellen Differentialgleichungen zweiter Ordnung $Hr + 2Ks + St + M + N(rt - s^2) = 0$. [Diss.] Leipzig, 1895. 8vo. 44 pp.

Riemann (B.). Sur les hypothèses qui servent de fondement à la géométrie. Traduit de l'allemand par J. Hoüel. 2e tirage. Paris, Hermann, 1895. 4to. Fr. 3.00

Sauvage (L.). Théorie générale des systèmes d'équations différentielles linéaires et homogènes. Paris, Gauthier-Villars, 1895. 4to. 184 pp. Fr. 6.00

Tannery (P.). See **Diophantus Alexandrinus.**

Tournois (A.). Leçons complémentaires d'algèbre et notions de géométrie analytique, à l'usage des candidats à l'École spéciale militaire de Saint-Cyr. Châteauroux, Majesté et Bourchardeau, 1895. 8vo. 208 pp. Fr. 4.50

Volkmann (P.). Franz Neumann. * 11. September, 1798, † 23. Mai 1895. Ein Beitrag zur Geschichte deutscher Wissenschaft. Dem Andenken an den Altmeister der mathematischen Physik gewidmete Blätter, unter Benutzung einer Reihe von authentischen Quellen gesammelt und herausgegeben. Leipzig, Teubner, 1895. 8vo. 8 and 68 pp., portrait. Mk. 2.40

Weber (E.). Applicazioni geometriche ed analitiche di calcolo differenziale ed integrale. Milano, Richiedi, 1895. 16mo. 264 pp. Fr. 3.50

ELEMENTARY MATHEMATICS.

Aicardi (V.). Il triangolo; nozione particola reggiata di questa figura seguita da 740 problemi risolti. Torino, 1895. 8vo. 206 pp. Fr. 4.00

Bakandall (G. A.). See **Harrison** (J.) and **Bakandall** (G. A.).

Beman (W. W.) and **Smith** (D. E.). Plane and solid geometry. Boston, Ginn, 1895. 12mo. 10 and 320 pp. $1.35

Bourdon. Éléments d'algèbre. 18e édition, revue et annotée par E. Prouhet. Paris, Gauthier-Villars, 1895. 8vo. 12 and 658 pp. Fr. 8.00

Diesener (H.). Die Stereometrie. Praktisches Unterrichtsbuch zur leichten Erlernung der Körperberechnung und der Verhältnisse der Linien und Flächen im Raum. 2te Auflage. Halle, Hofstetter, 1895. 8vo. 4 and 88 pp. Mk. 2.00

—— Die ebene Trigonometrie und die Goniometrie. Praktisches Unterrichtsbuch zur leichten Erlernung der Benennungen der trigonometrischen Funktionen, etc. 2te Auflage. Halle, Hofstetter, 1895. 8vo. 4 and 118 pp. Mk. 2.00

Dufailly (J.). Géométrie. 9e édition, conforme aux derniers programmes. Villefranche-de-Rouergue, Bardoux, 1895. 8vo. 432 pp. Fr. 5.00

Euclid. See **Taylor** (H. M.).

Ferrucci (A.). See **Serret** (G. A.).

Gambioli (D.). Raccolta di esercizi di aritmetica generale, algebra e meccanica elementare, con parecchi esempi già risolti; ad uso degli allievi dei licei, istituti technici, nautici e scuole militari. Bologna, Zanichelli, 1896. 16mo. 8 and 496 pp. Fr. 4.00

Gioffredo (A.). Sunti ragionati delle lezioni di aritmetica, geometria e sistema metrico, specialmente destinati alle scuole superiori, con note didattiche. Nuova edizione, riveduta ed ampliata. Saluzzo, Bodoni, 1896. 16mo. 64 pp. Fr. 0.50

Harrison (J.) and **Bakandall** (G. A.). Practical plane and solid geometry, with an introduction to the study of graphics. London and New York, Macmillan, 1895. 12mo. 192 pp. 2s. 6d.

Hoüel (J.). Tables de logarithmes à cinq décimales pour les nombres et les lignes trigonométriques, suivies de logarithmes d'addition et de soustraction ou logarithmes de Gauss, et de diverses tables usuelles. Nouvelle édition, revue et augmentée. Paris, Gauthier-Villars, 1895. 8vo. 48 and 118 pp. Fr. 2.00

—— See **Schrön** (L.) et **Hoüel** (J.).

Ingrami (G.). Elementi di algebra, ad uso delle scuole secondarie superiori. Bologna, Cenerelli, 1895. 8vo. 8 and 222 pp. Fr. 3.00

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2ND SERIES, VOL. II. No. 6.

MARCH, 1896.

NEW YORK:
PUBLISHED FOR THE SOCIETY
BY MACMILLAN & CO.
1896.

Entered at the Post-office at New York, N. Y., as second-class matter.
PUBLISHED MONTHLY, EXCEPT AUGUST AND SEPTEMBER.

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THE THREE GREAT PROBLEMS OF ANTIQUITY, CONSIDERED IN THE LIGHT OF MODERN MATHEMATICAL RESEARCH.

Vorträge über ausgewählte Fragen der Elementargeometrie. F. KLEIN. Ausgearbeitet von F. TÄGERT. Leipzig, Teubner, 1895. pp. 66.

AMONG the minor mathematical works published during the past year, one of the most interesting is Klein's Festschrift for the third meeting of the association for the advancement of mathematical and scientific teaching in the Gymnasia. The author has himself explained, in his paper *Ueber Arithmetisirung der Mathematik*, that his main object in writing this pamphlet was to emphasize the necessity for strict logical developments as a corrective to the tendency to rely too exclusively on intuitive proofs. Intended primarily for teachers of the more elementary mathematics, this pamphlet is confined to the three great problems of antiquity, as they appear in the light of modern research—or rather to the mathematical investigations for which these problems have furnished the text. These problems, (1) the duplication of the cube, (2) the trisection of an angle, (3) the quadrature of the circle, presented themselves at a very early stage in the development of mathematics, and they naturally present themselves correspondingly early in the mathematical development of the individual.

The Greek mathematicians, striving to solve these problems—problems for each of which the solution was impossible, within the domain of the geometry of the straight line and circle—were led to investigate and discover nearly all that lay within their originally unconsciously imposed boundaries. "Let it be granted, that a straight line may be drawn from any one point to any other point," "let it be granted, that a circle may be described from any centre, at any distance from that centre," these indicate the limitations of elementary geometry as understood by the Greeks, limitations recognized and formulated in the brightest period of Greek geometry by Euclid. It has been left to later times to recognize that other not less important limitations are implied in the definitions and axioms.

Still striving after the solution of these problems, Archytas, Eudoxus and their successors were led to enlarge their boundaries, admitting other methods, in the first instance the construction of conics, by which they solved the problems that they classed as "solid," and then the construction of higher curves, "linear loci," for use in treating those questions that were neither plane nor solid.

In this pamphlet the essential nature of the problems is sharply defined; the domain of Greek geometry is investigated, as to its contents and its boundaries; the correspondence with different parts of algebra is traced, and it is shown, in part all too briefly, how the successively enlarged domain of algebra corresponds with the different parts of geometry, the admissible algebraic operations being represented by the postulated geometrical constructions. Rational algebraic operations are represented by linear constructions; operations depending on the extraction of the square root can all be performed by means of straight lines and circles, in fact, by means of straight lines and a single circle, a possibility shown by Poncelet. These operations can also be performed by means of circles only, for Mascheroni wrote a book on geometry in which the only admissible construction is that of a circle, with any centre, passing through any assigned point. Klein refers to Hutt for an account of this book; a paper by Cayley in the *Messenger of Mathematics*, vol. 14, p. 179, may be more available for some readers. This paper gives sufficient account of the work to enable one to see the course of the proofs; the constructions depend chiefly on the conception of symmetry, points on a line being determined by the intersections of equal circles whose centres are symmetrically placed with respect to the line.

The pamphlet is divided into two parts, the first of which deals with algebraic numbers, the second with transcendental numbers. In the first part, Ch. I. explains the nature of a system of conjugate irrational quantities, and proves that there exists only one irreducible equation satisfied by one, and consequently by all, of the quantities of any such system. The chapter contains also a discussion of the nature of the algebraic equations that can be solved by the extraction of square roots; this discussion may be described as elementary, inasmuch as all the steps are minutely explained, but it gives the rigorous proof of the important theorem that an irreducible equation cannot be solved by the extraction of square roots, and hence cannot be constructed by means of straight lines and circles, unless its degree be a power of 2. In order to solve a cubic equation the domain of algebra must be enlarged by adjoining the operation of taking the cube root; hence problems whose algebraic expression leads to an equation of the third degree cannot be solved by elementary geometry. In Ch. II. it is shown that the first two of the three problems under discussion are of this type; they are therefore outside the domain of elementary geometry.

Since in Ch. I. the possibility of solving an apparently insoluble equation, if the degree be a power of 2, has presented itself, in Ch. III. this question is considered more carefully. It is not possible (*i. e.*, within the assigned limits of construction) to trisect *any* angle, but we can trisect a complete angle, we can divide the circumference of a circle into three equal parts. Can we divide the circumference into *any* number of equal parts? This inquiry opens out an extensive field of modern research, and may well serve to introduce the would-be mathematician to some comprehension of what modern mathematics means. The division into 3, 4 or 5 equal parts, with the bisection of each of these parts any number of times, gives the extent of the Greek achievement in this line, and was supposed to give the limit of all possible achievement. The list was however extended by Gauss, who showed that the division can be performed for any prime number of the form $2^{2^\mu}+1$, and for any number whose prime factors other than 2 are of this form, and non-repeated. Moreover, these numbers give all for which the division is possible.

In the first place, assuming for a moment that the prime must be of the form 2^h+1, it is shown that a necessary but not sufficient condition that this represent a prime is $h=$ a power of 2, 2^μ. It is remarked that the values 0, 1, 2, 3, 4, for μ give prime numbers 3, 5, 17, 257, 65537; that the values 5, 6, 7 for μ do not give prime numbers; and that the investigation has not been carried any further. At this point Klein suggests that possibly 4 is the highest value for μ that gives a prime. In the *Récréations mathématiques* of E. Lucas, however, the remark is made (vol. II., p. 235) that Eisenstein enunciated the theorem, "The number of primes of the form $2^{2^\mu}+1$ is infinite," giving no proof, though Lucas suggests that possibly he had one. The theorem may of course be purely conjectural.

It is now to be shown that the equation $\dfrac{z^p-1}{z-1}=0$,

i. e.
$$z^{p-1}+z^{p-2}+\cdots+z+1=0,$$

which with the ordinary use of the complex variable is the equation for p-section of a circle of unit radius, is irreducible. This, by the Gauss lemma, is made to depend on the proof that the expression cannot be resolved into factors with integral coefficients. Here Klein follows Eisenstein's method, which, while perfectly simple, is not satisfactory in point of form, for the proof is made to depend on the somewhat irrelevant transformation $z = x + 1$, by which it is shown that the contrary supposition is untenable. When it has been shown that the equation is irreducible the rather lengthy argument of Ch. I. shows that the degree of the equation, *i. e.*, $p-1$, must be a power of 2.

As regards the actual theorem, a different proof is to be found in the December number of the BULLETIN, in the paper by Dr. James Pierpont. This paper deals with the complete theorem under consideration, giving an admirably simple proof, which sets the essential part of the theorem in evidence. It is to be shown that the equation

$$F(x)=x^{(p-1)p^{a-1}}+\cdots+1=0,$$

where m, the number of parts into which the circle is to be divided, $= p^a$, cannot be rationally resolved; this is accomplished by following a method of Kronecker, preferable to that of Eisenstein, and presenting no greater difficulty; it is then shown that considering irrational factors of $F(x)$, the irrationalities depending only on the extraction of a square root, the only possible way in which $F(x)$ can split up the first time is into two conjugate factors of equal degree. Applying the theorem to each of these, and then to their factors, and so on, it is seen that the degree of F must be a power of 2.

The theorem having been established, the solution of the cyclotomic equation follows. It is possible to divide the roots into two periods whose sum and product are integral numbers; hence the two periods are the roots of a quadratic equation with integral coefficients. Taking one of the roots of this quadratic (that is, one of the two periods first formed), this can again be divided into two periods, given as the roots of a quadratic whose coefficients are rational in the roots of the first quadratic. In this way we obtain the solution of the cyclotomic equation by the solution of a chain of quadratics, the coefficients in any one of which depend on the roots of the preceding one. The process is fully explained in Ch. IV., being applied to the equation

$$z^{16}+z^{15}+\cdots+z+1=0,$$

from the solution of which there is obtained a linear construction for a regular polygon of 17 sides in a given circle.* The equations are so carefully worked out and so minutely explained that the chapter will throw much light on the difficulties that some readers will find in the first few pages of the paper. But seeing how much importance attaches to the fact that the equation is irreducible until the domain of rationality is extended, we could wish that Klein had thought it worth while to give the successive factors;

$$z^{16}+z^{15}+\cdots+1=\Big(z^8-\eta_0z^7+(2-\eta_0)z^6+(3-\eta_0)z^5+(1-2\eta_0)z^4+$$

$$(3-\eta_0)z^3+(2-\eta_0)z^2-\eta_0z+1\Big)\Big(z^8-\eta_1z^7 \text{ etc.},\cdots\Big),$$

$$z^8-\eta_0z^7+\cdots+1=\Big(z^4-\eta_0'z^3+\frac{\eta_0'\eta_0-\eta_1'+1}{2}z^2-\eta_0'z+1\Big)\Big(z^4-\eta_1'z^3$$

$$+\frac{\eta_1'\eta_0-\eta_0'+1}{2}z^2-\eta_1'z+1\Big),$$

$$z^4-\eta_0'z^3+\frac{\eta_0'\eta_0-\eta_1'+1}{2}z^2-\eta_0'z+1=\Big(z^2-\eta_0''z+1\Big)\Big(z^2-\eta_1''z+1\Big);$$

this would have thrown some additional light on the significance of the early part of the paper.

Chapter V. indicates briefly the points of interest in connection with the construction of higher equations. The cubic and biquadratic are solved by means of conics, and thus conics are needed for the solution of problems whose algebraic expression leads to such equations. An example that naturally presents itself is the construction of the regular heptagon by means of a parabola and a rectangular hyperbola; and it is at once perceived that if conics are admitted, many more regular polygons can be described. Apparently Descartes knew that the regular heptagon and nonagon can be described by conics, for when discussing constructions by curves of higher order, he says that these can be applied to construct regular polygons of 11 or 13 sides. Now the regular polygons with 3, 4, 5, 6, 8, 10, 12 sides were described by the Greeks, and supposing Descartes to be aware that for 7 and 9 sides only conics are needed, his idea was apparently that the first regular polygons for whose construction higher curves are required are those of 11 and 13 sides. (The 13-ic can however be described by conics.) The omission of any explicit mention of the 7-ic and 9-ic proves nothing in the case of Descartes, for at the point where he might naturally have given it he had already devoted several pages to the construction of cubic equations, and was probably, as he himself says in another place with less reason, "bored already with writing so much about it." Such omissions may surely be forgiven for the sake of his concluding remark: "Et j'espère que nos neveux me sauront gré, non seulement des choses que j'ai ici expliquées, mais aussi de celles que j'ai omises volontairement, afin de leur laisser le plaisir de les inventer."

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* In this connection it is perhaps worth while drawing attention to the very simple construction for the regular 17-ic given by Mr. H. W. Richmond in the *Quarterly Journal*, vol. 26, p. 206, 1893. The construction is simple in practice; theoretically it is not as simple as the one here given, for the equations are combined in such a way as to require circles in their construction. It might probably with slight modifications give a Mascheroni construction, which Klein remarks (p. 27) has not yet been considered for this case.

The first part of the paper has made clear the meaning of algebraic numbers, irrationalities successively introduced in the solution of algebraic equations. For the discussion of the third great problem, the quadrature of the circle, it is necessary to show that algebraic numbers form only a part, in point of fact an insignificant part, of the numbers that claim consideration. The chapter devoted to this cannot fail to interest many hitherto unfamiliar with Cantor's work. It is first to be shown that there is a (1, 1) correspondence between positive integers and all real algebraic numbers, that is to say, that the real algebraic numbers can be counted. To prove this it is shown, by a particular grouping of the irreducible equations whose roots give the algebraic numbers, that the numbers can be arranged in a singly infinite series of groups, those contained in any one group being finite in number; and that thus all the real algebraic numbers, positive and negative, can be arranged in an absolutely determinate order, which is not the order of magnitude; and so can be spoken of as 1st, 2nd, 3rd, etc., that is, put in (1, 1) correspondence with the positive integers.

The next step is to show how numbers can be constructed that shall not be contained in this orderly series. The number being required to lie within certain limits, so that there are given a certain number of decimal places, *e. g.*, 5, the digits in the following places have to be selected so that the number differs from all of the series. For a reason explained, the digit 9 is avoided. The 6th digit is chosen to be different from the 6th of the first algebraic number, and thus the number constructed will certainly be different from this;

the 7th digit is chosen to be different from the 7th of the second algebraic number, by which we ensure that the number written down is not the second, and so on. Hence we are assured of the existence of numbers that are not the roots of any algebraic equation; that is, the existence of transcendental numbers is proved, and it is shown how they can be written down. Moreover, since the choice of the digit to be written in any assigned place is restricted only by the exclusion of two digits, 9 and one other, we may choose any one of the 8 that are left, zero being admissible in the same way as any other. Hence between any two algebraic numbers there are 8^{∞} transcendental numbers, (not ∞^{8} as stated in the pamphlet,) and real algebraic numbers form only a small part of all numbers.

The possibility of determining whether a given number is transcendental or not cannot be considered generally. A terminating or circulating decimal is obviously algebraic; a non-terminating and non-circulating decimal obviously cannot be given in full. Transcendental numbers can therefore only be given by some law of formation, as is seen in the case of e and π; but since algebraic numbers can also be given in this way, the fact that a number is so given proves nothing. The impossibility of giving any general criterion is brought home by the nature of the proofs for e and π. The connection of the two numbers, exhibited in the fundamental equation $e^{i\pi}+1=0$, gives the transcendency of π as a corollary from the theorem:—The quantity e is not a root of any equation in which both coefficients and exponents are algebraic numbers. To the consideration of e and π chapters III. and IV. are devoted; these give the most recent proofs, published as late as 1893.

Since the number π is transcendental it follows that no algebraic curve can be found in which the relation of derived straight lines is that of the circumference of a circle to the diameter. Hence the quadrature of the circle, the third great problem of antiquity, cannot be solved by means of algebraic curves.

Thus in these few pages some of the most striking results of modern mathematics are made accessible to many who would otherwise hardly have heard of them. But while reading this brilliant exposition it is difficult to avoid cherishing a lurking regret, which is possibly very ungracious, that Klein could not himself spare time to arrange his work for publication; for though we have here in full measure the incisive thought and cultured presentation which together make even strict logic seem intuitive, yet at times we miss the minute finish and careful proportion of parts that we feel justified in expecting from him. And yet revision and consolidation might have seriously interfered with the graphic simplicity of these chapters, and left them less adapted to their special purpose. Any English-speaking association with aims similar to those of the German association for which the pamphlet was prepared would do a service in publishing a thoroughly good translation of this inspiring work and circulating it as widely as possible.

CHARLOTTE ANGAS SCOTT.

BRYN MAWR COLLEGE,
January 15, 1896.

PAINLEVÉ'S LECTURES ON DYNAMICS.

I. *Leçons sur l'intégration des equations différentielles de la Mécanique et Applications.* Par P. PAINLEVÉ. pp. 291; 4to (lithographed).

II. *Leçons sur le Frottement.* Par P. PAINLEVÉ. pp. VIII. + 111; 4to (lithographed). Paris, A. Hermann, 1895.

The publication of Mr. Painlevé's lectures on the integration of the differential equations of dynamics will be welcomed by everyone interested in the progress of theoretical mechanics. It is a long time since Jacobi's *Vorlesungen über Dynamik* appeared, and a strong need was felt of a systematic work which would contain the more recent researches in this important branch of mathematical science. Mr. Painlevé has admirably supplied this need. The lectures are not intended for beginners in theoretical mechanics, but rather as a supplementary course for those already familiar with its elements.

The first three lectures contain the fundamental definitions and principles of dynamics, the propositions relating to the first integrals of the motion of rigid systems and the theory of the motion of a solid body, the whole followed by a number of examples. Despite the brevity of this exposition it is clear and instructive, owing to several interesting remarks.

The fourth lecture deals with the general equations of the motion of systems. Lagrange's equations with the multipliers λ (also called Lagrange's equations of the first form) are derived from the consideration of the virtual work, which leads the author to a classification of systems into those with friction and without. In the next lecture this distinction is further discussed and a proof is given of Gauss's theorem that the constraint of a system is minimum when it is one without friction. Then the motion of systems with friction is studied and applied to the motion of a point on a surface. However the general theory of such systems constitutes the subject of a more detailed study in another of Mr. Painlevé's courses to which we will return later.

In the next two lectures, after establishing d'Alembert's equation and the principle of virtual velocities, the author passes to Lagrange's equations (of the second form), concluding with an enumeration of the principal systems without friction. All following lectures are restricted to such systems only.

Lectures VII.–IX. deal with the applications of Lagrange's equations, the cases being discussed separately when the equations of constraint do or do not contain the time explicitly. The author further proves Liouville's theorem relating to a large class of systems the motion of which can be found by quadratures. Several applications of this theorem are given. The classic problem of the motion of a point attracted by two fixed points and a number of other well chosen examples are treated in the course of these three lectures, the solutions being given with great simplicity and elegance.

Bour's differential equations of relative motion in the second Lagrangian form with examples taken from Gilbert's interesting memoir, on the applications of Lagrange's equations to the problem of relative motion, are the subject of the tenth lecture.

The next lecture begins with Lejeune-Dirichlet's theorem about the stability of the equilibrium of a system after which the author applies Lagrange's equations to the study of small oscillations of systems.

The remaining six lectures (XII.–XVII.) deal with the integration of the differential equations of dynamics principally after the works of Hamilton and Jacobi. The subjects of lectures XII. and XIII. are the canonic equations and Jacobi's theory of the last multiplier. Bour's canonic equations for relative motion are indicated but not discussed. The theory of the last multiplier is first applied to the motion of rigid systems in general. The application of the same theory to the canonic equations constitutes the subject of the fourteenth lecture, which is illustrated by a number of examples.

The next lecture contains Hamilton-Jacobi's method of integrating the differential equations of dynamics and the several simplifications which may occur in practice are discussed. A new demonstration is given of Liouville's theorem together with a generalization of the same due to P. Stäckel.

The sixteenth lecture presents the greatest novelty and interest. It deals with the more modern researches, the impulse to which was given first by Bertrand's principle of similitude in mechanics and quite recently by the remarkable paper of Stäckel (Crelle 1891) on the analytic equivalence of dynamic problems. The reader will find in this lecture a good deal of Mr. Painlevé's own researches chiefly contained in his important memoir on the transformation of the equations of Dynamics (J. de Liouville 1894). In the beginning of this lecture the author proves that the paths of a system depend in general on $2k$–1 arbitrary constants, k being the degree of freedom of the system, and he gives the condition necessary and sufficient in order that this number reduce to $2k$–2. The case is considered next, when these paths coincide with the geodesics. Then, after a discussion of the general case when the acting forces are independent of the velocities, the author passes to a detailed study of the real paths of a system; the principle of least action and Hamilton's principle are touched upon in the course of this lecture.

The last lecture relates to conservative systems. After establishing Poisson's theorem, the author returns once more to Hamilton-Jacobi's canonic equations and gives a concise but clear exposition of Jacobi and Mayer's methods for the integration of partial differential equations of the first order, concluding with the application to Jacobi's equation of Legendre's transformation, which is a particular case of Jacobi and Lie's transformations of contact.

It is to be regretted that the author did not add a chapter on the theory of perturbations which is so closely connected with the subject of his lectures. However, if these lithographed pages in which the exposition was necessarily limited are only the precursor of a more complete printed course, we may hope that the theory of perturbations will still find its place among Mr. Painleve's interesting lectures.

We now pass to the second of Mr. Painlevé's courses. The *lectures on friction* are a valuable contribution to rational mechanics. The author proposes to solve the following problem: *to develop for systems with friction a general theory similar to the one existing for systems without friction*, as far as the empiric laws of friction allow such a general treatment. The lectures before us deal first with discontinuous systems, and then with such continuous systems which can be considered as formed of infinitesimal elements that remain unchanged throughout the motion,* the other systems (to which belong extensible strings and membranes, compressible fluids and so on) being reserved by the author for another course to appear later. It must be remarked moreover that all cases where the conditions of constraint to which the systems are subject are over-abundant were systematically avoided by the author, since the motion of such systems in general depends on their interior constitution and belongs therefore to the theory of elasticity.

After recalling his general definition of systems with friction given in the fourth lecture on the integration of the differential equation of dynamics, derived from the consideration of the virtual work, the author examines first two special cases, namely, the motion of a point on a curve and on a surface, then passing to the general form of the laws of friction. We arrive here at a most interesting part of this course, both for its novelty and importance. The notion of partial friction is introduced and the author gives a detailed study of the laws of friction which result from the combination of the laws of partial friction (*combinaison des liaisons*). An important result is obtained when the conditions of constraint can be grouped into two parts: one, G_1, with friction and the other G_2, without. For, in such cases it is advantageous not to introduce the reactions due to the group G_1; to this purpose it suffices to form Lagrange's differential equations with regard to the parameters which would define the position of the system if only the conditions of constraint G_2 were taken into consideration.

After discussing the question of the compatibility of the conditions of constraint in general, the author classifies these conditions for continuous systems into three groups. To the first belong the conditions which are determined by a single relation between the parameters defining the position of the system. The author distinguishes nine types in this class. To the second group, consisting of five types, belong the conditions which are determined by two relations between the parameters defining the system. Finally, the conditions determined by three relations form the third group and are all conditions of constraint without friction.

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* To this class belong inextensible strings and membranes, incompressible fluids and so on. The position of such systems cannot be determined by means of a finite number of parameters. But, although Mr. Painlevé considers in his lectures only systems the position of which can be determined by means of a finite number of parameters, the general propositions developed by him can be immediately extended to the class here mentioned.

This general theory is followed by numerous applications worked out completely, and with the elegance and clearness which are characteristic of the two courses of lectures we have before us.

It is only fair in concluding this review to remark to Mr. Hermann's credit that the reading of these two volumes is not in the slightest degree trying to the eyes, which unfortunately could not be said with regard to, for instance, Mr. Klein's lithographed courses.

ALEXANDRE S. CHESSIN.

JOHNS HOPKINS UNIVERSITY,
January 24, 1896.

A GEOMETRIC PROOF OF A FUNDAMENTAL THEOREM CONCERNING UNICURSAL CURVES.

BY PROFESSOR W. F. OSGOOD.

1. IF $f(x,y)=0$ is the equation of an irreducible curve of deficiency 0, then, as is well known, the coördinates can be expressed as rational functions of a parameter λ:*

$$x=r_1(\lambda) \qquad y=r_2(\lambda)$$

where not only to a given value of λ corresponds one and only one point of the curve, but conversely to a given point (x,y) on the curve corresponds in general one and only one value of λ.† λ can be expressed as a rational function of x and y.

If a multiple-leaved Riemann surface spread out, say, over the x-plane be used to represent geometrically the above locus, $f(x,y)=0$, the deficiency of this surface will likewise be 0, and as is shown in the elements of Riemann's theory of functions,‡ there exist single-valued functions on such a surface having but a single pole, and that of the first order, and taking on every value once and only once on the surface. Call such a function λ. Being single-valued on the surface it will be a rational function of x and y: $\lambda=R(x,y)$,

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*SALMON, *Higher Plane Curves*, p. 30; CLEBSCH-LINDEMANN, *Geometrie* vol. I., p. 883.

† CAYLEY has given to such curves the name *unicursal*.

‡ KLEIN, *Modulfunctionen*, vol. I., p. 493 et seq. PICARD, *Traité d' analyse*, vol. II., Ch. XVI.

and hence through this equation the multiple-leaved surface is transformed conformally into the single-leaved λ-plane. Hence x is a rational function of λ. And similarly for y.

2. Conversely, let the equations

$$\left.\begin{array}{l} x=r_1(\lambda) \\ y=r_2(\lambda) \end{array}\right\} \qquad (1)$$

be given, where r_1 and r_2 are any rational functions of λ Then the locus of (x, y), if regarded as a plane curve, is an algebraic irreducible curve of deficiency 0; if represented by a multiple-leaved surface spread out over the x-plane, it gives rise to an algebraic surface consisting of a single piece, likewise of deficiency 0. (In connection with these statements see however the last lines of § 5.) But while to every value of λ there still corresponds one and only one point (x, y) of this surface, it is not true conversely that to a point (x, y) only one value of λ corresponds, as is shown by the example $x=\lambda^2$, $y=\lambda^2$. It is however always possible to replace λ by a new parameter μ, which is a certain rational function of λ:

$$\mu=r(\lambda)$$

such that x and y will be rational functions of μ, while to a given point (x, y) only one value of μ corresponds.*

3. The essential thought that underlies these theorems finds clear expression in the geometric method of conformal representation, and reciprocally the value of this method as a means of investigation is set in a strong light.

It is believed that it will not be without interest to the readers of the BULLETIN to see this simple example of the method worked out, and it is hoped furthermore that this example may be suggestive of a desirable form of presentation in instruction in the theory of functions.

4. Consider first the equation

$$x=r_1(\lambda)$$

To each value of λ corresponds one and only one value of x and the λ-surface consists of the single-leaved λ-plane. To each value of x correspond m values of λ, in general distinct, and the x-surface consists of an m-leaved Riemann surface connected through junctions along junction-lines. Call this surface T.

Now consider the second equation(1):

$$y=r_2(\lambda).$$

y will be a single-valued function on the surface T, since to every point of T corresponds only one value of λ. But it may happen that for *every* value of x two or more of the m corresponding values of y coincide; so that the surface T would not represent y as a function of x.

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* An algebraic proof of this latter theorem was given by Lüroth, *Math Annalen*, vol. 9, 1875.

Let us construct the surface that does represent y as a function of x. For this purpose first cut the leaves of T along the junction lines, thus obtaining m separate leaves, each containing one or more slits. Consider the first of these leaves. To each of its points corresponds a value of y, y_1, and it forms the first leaf of the surface to be constructed, S. Next let x leave this first leaf and crossing one of the slits, pass into a second leaf of T. Call the corresponding value of y in this second leaf y_2. Two cases can arise. Either y_2 is, in general, different from y_1 or y_2 is identical with y_1. In the first case this second leaf of T is to be connected with the first leaf along this slit and thus forms the second leaf of S. In the second case the two edges of the slit in the first leaf are to be joined with each other and this slit thus disappears from S;* the second leaf of T does not belong to S. The example already cited will serve here as illustration: $x=\lambda^2$, $y=\lambda^2$. The surface T consists of two leaves with junctions at 0 and ∞; the junction-line may be taken as the positive axis of reals. The points of the first leaf shall represent those pairs of values (x, λ) that flow continuously out of the pair $x=-1$, $\lambda=i$ when x describes this leaf; y_1 is thus equal to x. When x crosses the slit and thus passes into the second leaf of T, y_1 goes over into y_2. But $y_2=y_1$, both being equal to x, and hence in the construction of S this slit disappears from the first leaf. In this example this is the only slit in the first leaf and thus the surface that represents y as a function of x is the single-leaved x-plane.

Returning now to the general case, we treat the other slits in the first leaf, if there be any, in the same way and thus finally either all the slits are closed and the first leaf forms the whole surface S, or else this first leaf is connected with other leaves along at least one slit. Such leaves will surely belong to the surface S. These leaves may be already connected along their slits with the first leaf, as in the case

$$x=\lambda^6, \quad y=\lambda^3;$$

or perhaps it remained merely to join certain pairs of them along certain slits, as in the case

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* It may happen that a slit, beside abutting on the two junctions at its extremities, passes through one or more junctions in its course. In that case I shall consider each of the segments of the slit as a separate slit.

$$x=\lambda^6, \quad y=\lambda^2;$$

or it may be that one of these leaves contains a slit such that when the point (x, y) crosses this slit, y takes on still new values. Then these two leaves are to be joined along this slit and another leaf is added, as in the case

$$x=\lambda^6, \quad y=\lambda.$$

Proceeding in this way we finally obtain a closed surface S. This surface may or may not be the same as T. If it is, then it is of deficiency 0, being conformally related to the single-leaved λ-surface. λ is a single-valued function on this surface having no essentially singular points, and hence it is a rational function of x and y:*

$$\lambda=R(x, y).$$

Thus all the statements made in § 2 are proven.

5. If, however, only a part of the leaves severed from T have been incorporated into S, then it must be that in the construction of S as described above, a leaf L' was found in which the values of y were identical, for the same values of x, with values in another leaf L already belonging to S; and hence L' was not joined to S, the connection along the corresponding slit being between the leaves already in S.

Consider now the leaf L'. The values of y in this leaf are the same for the same values of x as in L. If we proceed to construct the Riemann surface for y as a function of x, starting with the leaf L', we shall obviously be led to a surface S' identical with S; thus S is repeated. If the leaves severed from T are not yet exhausted, it will be possible to construct from them still a third surface S'', likewise identical with S. And so on. Thus finally the leaves of T will have been combined in a set of ρ surfaces, $S, S', S'' \cdots S^{(\rho-1)}$, each identical with the first, S. ρ will divide m. Let $m=m'\rho$. Then m' is the number of leaves in S. The examples already cited illustrate clearly what has just been said.

So much for the surface T and the surfaces S. Let us turn now to the corresponding regions of the λ-plane. To each of the m leaves of the surface T corresponds a region of the λ-plane, regions corresponding to leaves that are connected bordering on each other. If we recall the way in which the S-surface was generated by the successive addition of new leaves, we see that the counterpart of this process in the λ-plane consists in starting with the region corresponding to the first leaf of S and then adding on bordering regions corresponding to the subsequent leaves of S, the final result being a connected region N, made up of m' of the former regions, which corresponds point for point and conformally to S. To S' will correspond a second such region N', to S'' a third, N'',...to $S^{(\rho-1)}$, $N^{(\rho-1)}$, and these ρ regions will just cover the λ-plane.

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* HARKNESS and MORLEY, *Theory of Functions*, § 174; KLEIN-FRICKE, *Modulfunctionen*, vol. I., p. 499.

Let λ_1 be any point lying in the region N. To λ_1 corresponds the point (x, y) of each of the surfaces $S, S', \cdots S^{(\rho-1)}$ and to these points correspond in turn values $\lambda_1', \lambda_1'', \cdots \lambda_1^{(\rho-1)}$ belonging respectively to $N', N'', \cdots N^{(\rho-1)}$. Let us from now on, discarding the surfaces $S', S'', \cdots S^{(\rho-1)}$, restrict ourselves to S, so that $\lambda'_1, \lambda_1'', \cdots \lambda_1^{(\rho-1)}$ will now be regarded as functions of (x, y) on S. Thus any region of S that does not contain any part of one of the slits that has been closed up or along whose edges the sheets have been connected differently in S from in T will be transformed conformally on a region of each of the domains $N, N', \cdots N^{(\rho-1)}$. If the point (x, y), moving on S, cross one of the slits just excepted, then, since this slit corresponds to the boundary between $N^{(i)}$ and $N^{(k)}$, $\lambda_1^{(i)}$ will cross this boundary and enter the region $N^{(k)}$, another one of the λ_1's entering at the same time the region $N^{(i)}$ just vacated by $\lambda_1^{(i)}$. And if (x, y) describe any closed path on S, the λ_1's will either describe closed paths or be permuted. Now form the symmetric product

$$\mu = \lambda_1 \lambda_1' \lambda_1'' \ldots \lambda_1^{(\rho-1)}. \qquad (3)$$

This function is in general monogenic on S, since this is true of each of the factors (on account of the conformal correspondence), and furthermore it is single-valued. It has no essentially singular point and is therefore a rational function of x and y,

$$\mu = R(x,y). \qquad (4)$$

The point $\lambda = \infty$ will lie in general entirely within one of the regions N, so that only one factor will ever become infinite, and this factor will afford a pole of the first order. No other factor becoming 0 simultaneously,* μ has but one pole and that of the first order, so that it takes on every value once and only once on S.

It thus appears that the surface S is transformed conformally on a single-leaved plane, the μ-plane. Hence we infer (1) that the deficiency of S is 0; (2) that x is a rational function of μ; for, because of the conformal relation between the surface S and the μ-plane, x behaves in every region of the μ-plane like a rational function; (3) that y is likewise a rational function of μ; for the Riemann surface that represents x as a function of y is conformally related to S and hence to the μ-plane, and y behaves therefore in every region of the μ-plane like a rational function; (4) that μ is a rational function of λ; this can also be inferred easily from the conformal representation, but as it follows immediately from equations (1) and (4), we will not insist on this method.

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* If these conditions, in a special case, should not happen to be fulfilled, λ may be replaced through a suitable linear transformation by $\bar{\lambda}$, so that they will now be fulfilled.

Thus all of the statements of § 2 have been proven.

Example.

$$x=\frac{(\lambda^4+\lambda^{-4}+14)^3}{108\ (\lambda^4+\lambda^{-4}-2)^2}=\frac{4\ (\mu^2-\mu+1)^3}{27\ (\mu-1)^2}$$

$$y=-\left(\frac{\lambda^2-1}{2\lambda}\right)^2=\mu$$

$$\mu=-\tfrac{1}{4}\,(\lambda-\lambda^{-1})^2$$

See Klein-Fricke, *Modulfunctionen*, vol. I. p. 75, for the division of the λ-plane. The $\rho=4$ regions N, N', N'', N''' appear in the figure on p. 80. The notation λ, μ is there just the reverse of that in this paper.

HARVARD UNIVERSITY,
CAMBRIDGE, MASS., December, 1895.

NOTES ON THE EXPRESSION FOR A VELOCITY-POTENTIAL IN TERMS OF FUNCTIONS OF LAPLACE AND BESSEL.

BY PROFESSOR JAMES MCMAHON.

1. *Differential equation for ψ.* The partial differential equation to be satisfied by a velocity-potential in an elastic fluid is, in rectangular coördinates,*

$$\frac{\partial^2\psi}{\partial x^2}+\frac{\partial^2\psi}{\partial y^2}+\frac{\partial^2\psi}{\partial z^2}=\frac{1}{a^2}\,\frac{\partial^2\psi}{\partial t^2},\tag{1}$$

in which a^2 = pressure/ density, and $\psi\ (x, y, z, t)$ is a function whose derivatives as to x, y, z give the velocity-components of the fluid particle that occupies the position (x, y, z) at time t.

2. *Particular solution in polar coördinates.* When (1) is transformed to polar coördinates r, θ, φ, it can, as shown in

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* RAYLEIGH, *Theory of Sound*, vol. II., p. 15.

standard works,* be satisfied by a product-function of the form

$$\psi_n = r^{-\frac{1}{2}} J_{\pm(n+\frac{1}{2})}(kr) . S_n(\theta, \varphi) . \genfrac{}{}{0pt}{}{\sin}{\cos} kat, \qquad (2)$$

in which S_n is any surface harmonic of order n, and $J_{\pm(n+\frac{1}{2})}$ is a Bessel function of order $\pm(n+\frac{1}{2})$. Here ψ_n stands for the part of ψ that involves nth order harmonics, and may be written in a more explicit form as the sum of the four terms implied in (2); thus

$$\begin{aligned} r^{\frac{1}{2}} \psi_n = J_m(kr) \left[S_n \sin kat + S'_n \cos kat\right] + \\ J_{-m}(kr) \left[S''_n \sin kat + S'''_n \cos kat\right], \end{aligned} \qquad (3)$$

where $m = n + \frac{1}{2}$, and the symbols S_n represent arbitrary nth order harmonics. The conditions of a particular problem may impose restrictions on these coefficients; and the chief object of this paper is to show how to choose the relations between the symbols S_n in order to adapt equation (3) to some of the typical problems in fluid motion to which the functions of Laplace and Bessel are applicable.† In all cases the complete value of ψ is to be built up by putting $n=0, 1, 2, \cdots$, and determining the arbitraries by the initial distribution of condensation and velocity.

3. *Phase angles.* It will be sometimes convenient to put equation (3) in the form

$$r^{\frac{1}{2}}\psi_n = T_n J_m(kr) \sin(kat + \epsilon_n) + T'_n J_{-m}(kr) \sin(kat + \epsilon'_n), \qquad (4)$$

wherein

$$T_n \cos \epsilon_n,\ T_n \sin \epsilon'_n,\ T_n \cos \epsilon'_n,\ T'_n \sin \epsilon'_n$$

take the place of S_n, S'_n, S''_n, S'''_n; the phase angles ϵ_n, ϵ'_n, being thus functions of θ, φ.

4. *Lemma.* To find the relations between the phase angles ϵ_n, ϵ'_n, and between the coefficients T_n, T'_n, in order that the motion of the medium at a great distance from the origin may be approximately represented by a single wave, either diverging from the pole or converging towards it:

When r is large put

$$J_{\pm m}(kr) = r^{-\frac{1}{2}} \cos(kr - \tfrac{1}{4}\pi \pm \tfrac{1}{2} m\pi), \ddagger$$

dropping a numerical factor; substitute in (4), and change trigonometric products into sums, then

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*GRAY and MATTHEWS, *Bessel Functions*, pp. 213, 215. *Theory of Sound*, Vol. II., pp. 205, 229, 230, in which the J functions are replaced by two other particular solutions of the equation that $r^{-\frac{1}{2}}J$ satisfies.

† The advantage of the form of solution in equation (3) is that it gives a "realized" result in terms of known functions and without restriction of phase.

‡ Bessel Functions, p. 40.

$$\begin{aligned} r\psi_n = {} & T_n \sin\left[k(at+r)+\epsilon_n-\tfrac{1}{4}\pi-\tfrac{1}{2}m\pi\right]+ \\ & T'_n \sin\left[k(at+r)+\epsilon'_n-\tfrac{1}{4}\pi+\tfrac{1}{2}m\pi\right] \end{aligned} \tag{5}$$

$$+ \text{ terms involving } at-r.$$

This expresses the superposition of a divergent and a convergent wave, represented by the terms in $at-r$, $at+r$, respectively. In order that the latter wave may be absent, the corresponding terms indicated in (5) must cancel each other for all values of the variables. This requires that $T_n = T'_n$, and also that

$$(\epsilon'_n+\tfrac{1}{2}m\pi)-(\epsilon_n-\tfrac{1}{2}m\pi) = \text{an odd multiple of } \pi,$$

say congruent with $(2n+1)\pi$, *i. e.*, with $2m\pi$,

$$\therefore \epsilon'_n-\epsilon_n = m\pi, \text{ or a congruent.} \tag{6}$$

Similarly the divergent wave is absent when

$$T_n = T'_n, \quad \epsilon_n-\epsilon'_n = m\pi.$$

Thus the velocity potential of a wave which at a great distance from the origin is $\left\{\begin{array}{l}\text{divergent}\\ \text{convergent,}\end{array}\right.$ consists of terms of the form

$$r^{\frac{1}{2}}\varphi_n = T_n[J_m(kr)\sin(kat+\epsilon_n)+J_{-m}(kr)\sin(kat+\epsilon_n\pm m\pi)]. \tag{7}$$

5. *Relations between the symbols S_n, for divergence or convergence.* Put, according to Art. 3,

$$S_n = T_n \cos\epsilon_n, \qquad S'_n = T_n \sin\epsilon_n,$$
$$S''_n = T_n \cos(\epsilon_n\pm m\pi), \quad S'''_n = T_n \sin(\epsilon_n\pm m\pi),$$

whence the required relations are

$$\therefore \quad S''_n = \mp S'_n \sin m\pi, \quad S'''_n = \pm S_n \sin m\pi. \tag{8}$$

6. *Resolution of ψ_n.* The expression in (3) can be exhibited as the sum of two parts representing divergent and convergent waves. For write

$$\begin{aligned} r^{\frac{1}{2}}\psi_n = {} & T_n[J_m(kr)\sin(kat+\epsilon_n)+J_{-m}(kr)\sin(kat+\epsilon_n+m\pi)]+ \\ & T'_n[J_m(kr)\sin(kat+\epsilon'_n)+J_{-m}(kr)\sin(kat+\epsilon'_n-m\pi)], \end{aligned} \tag{9}$$

the first part representing a divergent wave, and the latter a convergent one, by (7); compare with (3), equate coefficients, and solve; then T_n, T'_n, ϵ_n, ϵ'_n are determined by

$$2\,T_n\cos\epsilon_n = S_n+S'''_n\sin m\pi, \; 2\,T_n\sin\epsilon_n = S'_n-S''_n\sin m\pi,$$
$$2\,T'_n\cos\epsilon'_n = S_n-S'''_n\sin m\pi, \; 2\,T'_n\sin\epsilon'_n = S'_n+S''_n\sin m\pi. \tag{10}$$

7. *Application to spherical vibrator.* The surface of a sphere of radius c is maintained in vibration with given frequency, and the normal surface-velocity is a given harmonic function of latitude and longitude of the form

$$v_n(\theta, \varphi, t) = \bar{S}_n \sin kat + \bar{S}'_n \cos kat, \tag{11}$$

to find the velocity potential at any point (r, θ, ψ) of the medium at any time t:

Use (3), find the radial velocity-function $\delta\psi_n/\delta r$, and put $r = c$, then

$$\begin{aligned} v_n = \left(\frac{\delta \psi_n}{\delta r}\right)_{r=c} &= F_m\,(kc)\;[S_n \sin kat + S'_n \cos kat] + \\ &\quad F_{-m}\,(kc)\;[S''_n \sin kat + S'''_n \cos kat], \end{aligned} \tag{12}$$

in which $F_m\,(kr) = \frac{\delta}{\delta r}\,.\;\; r^{-\frac{1}{2}}\, J_m\,(kr)$.

Identifying (11),(12) gives two equations, and the conditions for divergence in (8), furnish two others to determine S_n, S'_n, S''_n, S'''_n in terms of $\bar{S}_n$, $\bar{S}'_n$; thus

$$\begin{aligned} \gamma\, S_n &= \alpha\, \bar{S}_n + \beta\, \bar{S}'_n, \quad S''_n = -S'_n \cos n\pi, \\ \gamma\, S'_n &= -\beta\, \bar{S}_n + \alpha\, \bar{S}'_n, \quad S'''_n = S_n \cos n\pi, \end{aligned} \tag{13}$$

where $\alpha = F_m\,(kc)$, $\beta = F_{-m}\,(kc) \cos n\pi$, $\gamma = \alpha^2 + \beta^2$; and these values of the symbols S_n substituted in (3) give the required velocity-potential at an external point.*

The internal potential is of different form; for the disturbed region now includes the point $r = 0$, at which $J_{-m}\,(kr)$ becomes infinite, hence the arbitrary coefficients of this function must be zero. Putting $S''_n = S'''_n = 0$, and identifying (11), (12), determines S_n, S'_n, and gives

$$r^{\frac{1}{2}}\psi_n = \frac{1}{F_m\,(kc)}\, J_m\,(kr) \left[S_n \sin kat + \bar{S}'_n \cos kat \right]. \tag{14}$$

8. *Free vibrations inside a fixed spherical envelope.* Here, as in the latter part of Art. 7, the form is

$$r^{\frac{1}{2}}\psi_n = J_m(kr) \left[S_n \sin kat + S'_n \cos kat \right], \tag{15}$$

but the possible values of the free periods, which depend on k, are to be found from the equation $F_m(kc) = 0$, *i. e.*

$$\frac{\delta}{\delta r}\left[r^{-\frac{1}{2}} J_m(kr) \right]_{r=c} = 0;\dagger \tag{16}$$

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* This solution differs from that given in Theory of Sound, vol. II., p. 207, by being expressed in "realized" form by means of Bessel functions, and by not restricting v_n to be in the same phase over the surface $r = c$.

† *Annals of Mathematics*, vol. 9, No. 1, p. 27.

while the symbols S_n are determined by the harmonic elements of the initial distribution of velocity and condensation.*

9. *Free vibrations between two concentric spherical surfaces.* Since the radial velocity at the surface $r = r_1$ is zero, then

$$F_m(kr_1)\Big[S_n \sin kat + S'_n \cos kat\Big] + F_{-m}(kr_1)\Big[S''_n \sin kat + S'''_n \cos kat\Big] = 0;$$

and there is a similar equation involving r_2.

These must be satisfied for all values of $\theta, \varphi, t,$

$$\therefore S_n F_m(kr_1) = -S'_n F_{-m}(kr_1);\ S'_n F_m(kr_1) = -S''_n F_{-m}(kr_1),$$

with two similar equations in r_2,

$$\therefore \frac{S''_n}{S_n} = \frac{S'''_n}{S'_n} = -\frac{F_{-m}(kr_1)}{F_m(kr_1)} = -\frac{F_{-m}(kr_2)}{F_m(kr_2)} = \rho, \text{ say.} \qquad (17)$$

The possible values of k, and of the wave length $2\pi/k$, are to be found from the third of these equalities;† and then S''_n, S'''_n are known multiples of S_n, S'_n. Thus (3) takes the form

$$r^{\frac{1}{2}}\psi_n = [J_m(kr) + \rho J_{-m}(kr)]\ (S_n \sin kat + S'_n \cos kat), \qquad (18)$$

an equation which, extended to the whole of space, gives a series of nodal spherical surfaces, of which $r = r_1$, and $r = r_2$ are a pair. At such surfaces the superposed divergent and convergent waves interfere.

ADDITIONAL NOTE ON DIVERGENT SERIES.

BY PROFESSOR A. S. CHESSIN.

In a previous note (pp. 72–75) it has been shown that every divergent series oscillating between finite limits can by a proper arrangement of its terms be made convergent. We will now extend those results to the case when one or both limits between which the series oscillates are infinite. To this end it suffices to consider, together with *regular* sequences of numbers, such sequences as tend to infinity *in a determinate way.* The number N_i then may be $+\infty$ or $-\infty$ and the infinite series

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* The work is exemplified for the case $n = 1$, *Theory of Sound*, pp. 236, 237.

† *Annals of Mathematics*, vol. 9, No. 1, pp. 29, 30.

$$a_{1,i}+a_{2,i}+\cdots+a_{n,i}+\cdots$$

of Theorem I. (p. 73) then tends to infinity *in a determinate way.* This series is *unconditionally divergent.* A similar change in the considerations of Theorem II. (p. 74) shows that the series

$$(u_1+u_2+\cdots+u_{\mu_{1,i}})+(u_{\mu_{1,i}+1}+\cdots+u_{\mu_{k,i}})+\cdots$$

tends to infinity in a determinate way, but it must be remembered that here each expression in parenthesis is to be considered as a *single* term, and that this determinateness would be lost at once if the brackets were dropped.

Example: $1-2+3-4+5-\cdots$

$$g=2;\quad N_1=+\infty=1+(-2+3)+(-4+5)+\cdots$$
$$N_2=-\infty=(1-2)+(3-4)+\cdots$$

We will say that an infinite series tends to infinity only if it does so in a determinate way. It was also in this sense that the expression was used in Theorem III. (p. 75). We may now add

Theorem VI. Every oscillating series can by a proper arrangement of its terms be made convergent or tending to infinity.

To conclude, a remark must be made with regard to the different rearrangements of the terms of an infinite series. On p. 75 it was mentioned that by a proper arrangement, both commutative and associative, of the terms of a conditionally divergent series, this series can be made to converge to any arbitrarily assigned number. It is important to take into consideration whether the associative change precedes the commutative or whether the inverse takes place. Of course the remark just mentioned applies only to series such that $\lim\limits_{m=\infty}(u_m)=0$, if u_m be the general term of the series, this condition being implicitly assumed in the extension of Riemann's proposition to conditionally divergent series.

Consider now a divergent series as given in Theorem II. (p. 74) and let again

$$N_i=(u_1+\cdots+u_{\mu_{1,i}})+(u_{\mu_{1,i+1}}+\cdots+u_{\mu_{k,i}})+\cdots$$

Here each expression in parenthesis is to be regarded as a *single* term, and the series may be either absolutely convergent or only semi-convergent. It seems, therefore, that we were not justified in saying (Theorem V.) that every conditionally divergent series can by a proper arrangement of its terms be made *semi-convergent;* that we should have used the word *convergent* instead. But the character of semi-convergence appears at once if we reverse the order of our changes in the original series, *i. e.*, if we first introduce a commutative and then proper associative changes. An oscillating series having necessarily an infinite number of positive and negative terms, the numbers N_i of Theorem II. will in general change their values if the order of terms in the given oscillating series be changed. Indeed, provided $\lim\limits_{m=\infty}(u_m)=0$, it is possible by a proper commutative arrangement to make not only the N_i but also their number g quite arbitrary. This is why we have used the word semi-convergent, and not convergent, in Theorem V.

As an example, let us take again the series (2) of the first note:

$$\tfrac{1}{2}-\tfrac{2}{3}+\tfrac{2}{3}-\tfrac{3}{4}+\tfrac{3}{4}-\cdots$$

which may be written as follows:

$$\sum_{1}^{\infty}\left\{\frac{2n-1}{2n}-\frac{2n}{2n+1}+\frac{2n}{2n+1}-\frac{2n+1}{2n+2}\right\}$$

If we now introduce the following communicative change

$$\sum_{1}^{\infty}\left\{\frac{2n-1}{2n}-\frac{2n}{2n+1}-\frac{2n+1}{2n+2}+\frac{2n}{2n+1}\right\}$$

we obtain $g=3$ and

$$N_1=\tfrac{1}{2}=\tfrac{1}{2}-\sum_{1}^{\infty}\left\{\frac{2n}{2n+1}+\frac{2n+1}{2n+2}-\frac{2n}{2n+1}-\frac{2n+1}{2n+2}\right\}$$

$$N_2=-\tfrac{1}{2}=\sum_{1}^{\infty}\left\{\frac{2n-1}{2n}-\frac{2n}{2n+1}-\frac{2n+1}{2n+2}+\frac{2n}{2n+1}\right\}$$

$$N_3=-\tfrac{3}{2}=\tfrac{1}{2}-\tfrac{2}{3}-\tfrac{3}{4}+\sum_{1}^{\infty}\left\{\frac{2n}{2n+1}+\frac{2n+1}{2n+2}-\frac{2n+2}{2n+3}-\frac{2n+3}{2n+4}\right\}$$

Thus the particular change here introduced has increased the number g by unity. It is easy to apply a number of other modifications by following the same process.

NOTES.

At the Lübeck meeting of the German Mathematical Society, held in September, 1895, it was decided to combine in one volume the official reports of the Vienna and Lübeck meetings. This will form vol. IV of the *Jahresbericht der Deutschen Mathematiker-Vereinigung* (Berlin, Reimer). Through the courtesy of the executive committee we have received advance sheets of the first twenty pages of this volume, containing the general reports of the secretaries, but not the abstracts of the papers.

It appears from the general report of the Vienna meeting (September, 1894) that the following papers should be added to the partial list given in the BULLETIN OF THE AMERICAN MATHEMATICAL SOCIETY, vol. I. (1894-95), pp. 78-79 :

GUTZMER (Berlin): "A new derivation of Kirchhoff's expression for Huygens's principle."

KIEPERT (Hannover): "On the mathematical preparation desirable for specialists in assurance."

KRAUSE (Dresden): "On the theory of the transformation of the elliptic functions."

LANDSBERG (Heidelberg): "On the theory of algebraic numbers."

LERCH (Prague): "On an integral occurring in Cauchy's transformation of the elementary elliptic function of the third kind."

J. SCHMIDT (Budapest): "Remarks on Bolyai's Tentamen."

WÄLSCH (Prague): "On a method of treating surfaces of the third order."

WANGERIN (Halle): "On the history of the theory of conformal transformation."

THE papers originally announced by Nagy, Peschka, Schapira, Schlesinger, and Studnička (loc. cit.) were not read.

At the Lübeck meeting the following papers were read:

HILBERT (Göttingen) and MINKOWSKI (Königsberg) : Account of their special report on the present state of the theory of numbers.

E. KÖTTER (Berlin): Account of his special report on the development of synthetic geometry.

FREGE (Jena): "On Prof. Peano's and my own symbolic algebra (Begriffsschrift)."

WANGERIN (Halle): "On Franz Neumann's mathematical works."

Lampe (Berlin): "On the compilation of a general bibliography."

Heffter (Giessen): "On common divisors and common multiples of linear differential expressions."

Voss (Würzburg): "On infinitesimal deformations of surfaces."

Pokrovsky (Kiev): "On the hyperelliptic functions of two arguments."

Souslov (Kiev): "On a continuous group of Darboux rotations."

Jukovsky (Moscow): "Geometrical interpretation of S. Kovalevsky's case of the motion of a heavy rigid body about a fixed point."

Fricke (Braunschweig): "The regions of discontinuity of the groups of real linear substitutions of a complex variable."

Klein (Göttingen): "On the theory of ordinary continued fractions."

Gordan (Erlangen): "Pascal's theorem."

Schubert (Hamburg): "Correlative correspondence in n dimensions."

Gutzmer (Berlin): "On certain linear differential equations."

Godt (Lübeck): "On the Feuerbach circle and the Steiner curve of fourth order and third class."

Kohn (Vienna): "On the geometric interpretation of homogeneous coördinates."

London (Breslau): "On cubic constructions."

Meyer (Clausthal): "On L. Schendel's memoir on infinite series and products."

Gutzmer (Berlin): "On Schoute's paper on certain envelopes."

Schütz (Göttingen): "On an interrelated group of thermodynamic, electrodynamic and astrophysical facts."

Sommerfeld (Göttingen): "Diffraction problems in exact treatment."

The special report on the present state of the theory of numbers, by Hilbert and Minkowski, will probably be published before the end of the present year; E. Kötter's special report on the development of synthetic geometry is expected to follow soon.

Plans for a number of other reports on special branches of mathematics were discussed at both meetings. Thus Pringsheim is preparing a report on the theory of infinite series, Czuber on the calculus of probabilities, Stäckel on differential geometry, Wälsch on line geometry. It was resolved at Lübeck that Wirtinger's report on Abelian functions should be postponed till after the publication of the third volume of Weierstrass' collected works and of his lectures on Abelian functions. At the same time, the theory of the functions of a real argument was designated as a subject on which a report would be desirable.

The project of a mathematical dictionary has taken somewhat more definite form, the academies at Vienna, Munich and Göttingen having promised financial assistance; B. G. Teubner will be the publisher.

The Society expressed its sympathy with the idea of an International Congress of Mathematicians, but declined to take the initiative in the matter. A communication from Prof. Molenbroek (Hague), concerning the formation of an international association for the promotion of the study of quaternions and allied subjects, was not received very favorably, the Society deeming it unwise to form such an association merely for promoting a very restricted branch of mathematical science.

Both at Vienna and Lübeck the German Mathematical Society held its sessions together with the section for mathematics and astronomy of the Society of German Naturalists and Physicians. At Lübeck a combined session with the physicists was arranged for. Besides the last two of the above-named papers, a report by Prof. HELM (Dresden), "On the present state of energetics," was read at this session. It provoked a prolonged and thorough discussion, but will not be included in the *Jahresbericht*.

A committee, consisting of Professors Brill, Wangerin and Gutzmer, was appointed to draft and present an address of congratulation to Prof. Weierstrass on his eightieth birthday (31st October, 1895).

The officers of the Society are at present: A. Brill, President; A. Gutzmer, Secretary and Treasurer; A. Wangerin and A. Gutzmer, Publication Committee; F. Klein and G. von Escherich, members of the Council. The membership is 273.

AMONG the prizes awarded at the annual meeting of the French Academy of Sciences were the following: *Prix Francœur* (1,000 francs), awarded annually for discoveries or works useful to the progress of mathematical science, to J. Andrade; *Prix Poncelet* (2,000 francs), awarded annually for the work, published during the preceding ten years, which, in the judgment of the Academy, has been most useful to the progress of mathematical science, to G. Robin for the whole of his work on mathematical physics; *Prix Petit d'Ormoy* (10,000 francs), awarded every second year, for works on pure or applied mathematics, to the late Albert Ribaucour.

The subjects set by the Academy for the next mathematical prizes are as follows: *Grand Prix des Sciences mathématiques* (10,000 francs), to advance in some important respect the algebraic theory of groups of substitutions of n letters, papers to be submitted by October 1, 1896. *Prix Bordin* (3,000 francs), to advance in some important respect the theory of geodesic lines, the case of a linear element with any number of variables not being excluded, papers to be submitted by October 1, 1896. *Prix Damoiseau* (1,500 francs); subject in 1896, to connect, by means of the theory of perturbations, the different appearances of Halley's comet, going back to Toscanelli's observations in 1456 and taking into account the attraction of Neptune, and to calculate with precision the next return of this comet in 1910; subject in 1898, to give an exposition of the theory of the perturbations of Saturn's satellite Hyperion, discovered in 1848 simultaneously by Bond and Lassell, taking into account principally the action of Titan, to compare the observations with the theory, and to deduce therefrom the value of the mass of Titan. Papers for the *Prix Damoiseau* must be submitted by June 1.

THE UNIVERSITY OF CHICAGO: SUMMER 1896. The following mathematical courses will be offered: By Professor Moore, Theory of Numbers, Differential Equations (with introduction to Lie's continuous transformation-groups); by Professor Bolza, Theory of Substitutions, Theory of Functions of a Complex Variable; by Professor Miller, of the University of Indiana, Analytic Geometry of Three Dimensions; by Dr. Young, Conferences on Mathematical Pedagogy, Theory of Equations, College Algebra; by Mr. Slaught, Advanced Integral Calculus, Introductory Course in Differential and Integral Calculus; and by Mr. Baker, Analytic Geometry of the plane. The pedagogical conferences are two hours weekly for six weeks, and the other courses are four or five hours weekly for twelve weeks from July 1, 1896. Those who expect to work in mathematics at the University of Chicago during the coming summer, as as well as those who desire further information, are requested to communicate with Professor Moore.

THE 1896 volume of the *Annuaire*, published by the Bureau des Longitudes, has been issued. It contains the usual mass of material devoted to astronomical and other science. Among the appendices are two articles by M. Cornu, upon forces acting at a distance, and upon Fresnel's optical work. There is also a report by M. Janssen about his third ascent of Mont Blanc. The list of members of the Bureau contains the names of two Americans: Dr. B. A. Gould and Mr. G. Davidson.

NEW PUBLICATIONS.

I. HIGHER MATHEMATICS.

Abel (N. H.). Untersuchungen über die Reihe $1 + \frac{m}{1} x + \frac{m(m-1)}{1.2} x^2 + \frac{m(m-1)(m-2)}{1.2.3} x^3 + \ldots$ (1826.) Herausgegeben von A. Wangerin. (Ostwald's Klassiker der exakten Wissenschaften, No. 71.) Leipzig, Engelmann, 1896. 8vo. 46 pp. Cloth. Mk. 1.00

Coculesco (N.). Sur les expressions approchées des termes d'ordre élevé dans le développement de la fonction perturbatrice. Paris, Gauthier-Villars, 1895. 4to. 90 pp.

Conant (L. L.). The number concept; its origin and development. New York, Macmillan, 1896. 12mo. Cloth. $2.00

Dewey (J.). See McLellan (J. A.) and Dewey (J.).

Epstein (P.). Zur Lehre von den hyperelliptischen Integralen. [Diss.] Strassburg, 1895. 4to. 58 pp.

Göpel (A.). Entwurf einer Theorie der Abel'schen Transcendenten erster Ordnung. (1847.) Herausgegeben von H. Weber. Aus dem Lateinischen übersetzt von A. Witting. (Ostwald's Klassiker der exakten Wissenschaften, No. 67.) Leipzig, Engelmann, 1896. 8vo. 60 pp. Cloth. Mk. 1.00

Klein (F.). Ueber die Arithmetisirung der Mathematik. [*Nachrichten der K. Gesellschaft der Wissenschaften zu Göttingen*, 1895, Heft 2.] 8vo. 10 pp.

———. Ueber eine geometrische Auffassung der gewöhnlichen Kettenbruchentwickelung. [*Nachrichten der K. Gesellschaft der Wissenschaften zu Göttingen*, 1895, Heft 3.] 8vo. 3 pp.

McLellan (J. A.) and Dewey (J.). The psychology of number and its applications to the methods of teaching arithmetic. (International education series, No. 23.) New York, Appleton, 1895. 8vo. 12 and 310 pp. Cloth. $1.50.

Matthiessen (L.). Grundzüge der antiken und modernen Algebra der litteralen Gleichungen. (1878.) 2te Ausgabe (unchanged, but reduced in price). Leipzig, Teubner, 1896. 8vo. 16 and 1,002 pp. Mk 8.00

Molk (J.). See Tannery (J.) and Molk (J.).

Neppi-Modona (A.) e Vannini (T.). Questioni e formole di geometria analitica ad una e due dimensioni. Palermo, 1895. 8vo. 320 pp. Fr. 7.50

Pascal (E.). Teoria delle funzioni ellittiche. Milano, Hoepli, 1896. 16mo. 12 and 228 pp.

Poincaré (H.). Calcul des probabilités. Leçons professées au cours de physique mathématique pendant le 2e semestre 1893–94, rédigées par A. Quiquet. Paris, Carré, 1896. 8vo. 280 pp.

Puchberger (E.). Eine allgemeinere Integration der Differentialgleichungen. Heft 3. Wien, Gerold, 1896. 8vo. 6 and 52 pp. Mk. 1.60.

Tannery (J.). et **Molk** (J.). Éléments de la théorie des fonctions elliptiques. (En 4 volumes.) Vol. II: Calcul différentiel, partie 2. Paris, Gauthier-Villars, 1896. 8vo. 6 and 300 pp. Fr. 9.00

Torres (L.). Memoria sobre las maquinas algébricas. Madrid, 1895. 4to. 108 pp. Fr. 4.00

Vannini (J.). See **Neppi-Modona** (A.) e **Vannini** (J.).

Velde (A.). Ueber die Curven, deren Bogen der Tangente des Lichtstrahlwinkels proportional ist, und die damit verwandten Curvenscharen. [Diss.] Halle, 1895. 8vo. 42 pp., 2 plates.

Wangerin (A.). See **Abel** (N. H.).

Weber (H.). See **Göpel** (A.).

Witting (A.). See **Göpel** (A.).

II. ELEMENTARY MATHEMATICS.

Esmarch (B. K.). Die Kunst des Stabrechnens. Gemeinfassliche und vollständige Anleitung zum Gebrauche des Rechenstabes auf allen Gebieten des praktischen Rechnens. Leipzig, Günther, 1896. 8vo. 192 pp., 2 plates. Cloth. Mk. 4.00

Gelin (E.). Précis de trigonométrie rectiligne. Namur, 1895. 8vo. 72 pp. Fr. 1.50

Holman (S. W.). Computation rules and logarithms, with tables of other useful functions. New York and London, Macmillan, 1896. 8vo. 14 and 74 pp. Cloth. $1.00

Kröger (M.). Leitfaden für den Geometrie-Unterricht. 10te Auflage. Hamburg, Meissner, 1895. Mk. 1.00

Lodge (A.). Mensuration for senior students. London and New York, Longmans, 1895. 12mo. 14 and 274 pp. $1.50

Meigen (F.). Lehrbuch der Geometrie. (Technische Lehrhefte, Mathematik, Heft 4.) Hildburghausen, Pezoldt, 1895. 8vo. 4 and 82 pp. Mk. 2.00

Nell (A. M.). Fünfstellige Logarithmen der Zahlen und der trigonometrischen Functionen, nebst den Logarithmen für Summe und Differenz zweier Zahlen, deren Logarithmen gegeben sind, sowie einigen anderen Tafeln. 8te Auflage. Darmstadt, 1895. 8vo. 20 and 104 pp. Cloth. Mk. 1.50

PINCHERLE (S.). Algebra elementare. 6a edizione. Milano, Hoepli, 1896. 16mo. 8 and 210 pp.

——— Esercizî sull' algebra elementare. Milano, Hoepli, 1896. 24mo. 135 pp.

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Whole No. 46.

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Articles for insertion should be addressed to the

BULLETIN OF THE AMERICAN MATHEMATICAL SOCIETY,

41 East 49th Street, New York City.

The following dates have been fixed for the meetings of the Society:

	P. M.		P. M.
Sat., Oct. 26, 1895,	3:00	Sat., Feb. 29, 1896,	3:00
" Nov. 30, "	"	" Mar. 28, "	"
Fri., Dec. 27, "	"	" April 25, "	"
Sat., Jan. 25, 1896,	"	" May 23, "	"

[illegible]

16

BULLETIN

OF THE

AMERICAN MATHEMATICAL SOCIETY.

CONTINUATION OF THE BULLETIN OF THE NEW YORK MATHEMATICAL SOCIETY.

A HISTORICAL AND CRITICAL REVIEW
OF MATHEMATICAL SCIENCE.

EDITED BY
THOMAS S. FISKE, ALEXANDER ZIWET, FRANK MORLEY.

2ND SERIES, VOL. II. NO. 7.

APRIL, 1896.

NEW YORK:
PUBLISHED FOR THE SOCIETY
BY MACMILLAN & CO.
1896.

Entered at the Post-office at New York, N. Y., as second-class matter.
PUBLISHED MONTHLY, EXCEPT AUGUST AND SEPTEMBER.

Whole No. 47. $5.00 a year.

A TWO-FOLD GENERALIZATION OF FERMAT'S THEOREM.

Presented to the American Mathematical Society, February 29, 1896.

BY PROFESSOR ELIAKIM HASTINGS MOORE.

Formulation of the generalized Fermat theorem III $[k+1, n; p]$. §§ 1–4.

1. In *Gauss's* congruence notation *Fermat's theorem* is:

$$\mathrm{I}_1 \qquad a^p - a \equiv 0 \pmod p$$

where p is any prime and a is any integer:
or, otherwise expressed,

I_2 *The two rational integral functions of the indeterminate X with integral coefficients*

$$X^p - X\,, \quad \prod_{a=0}^{a=p-1}(X+a)$$

are identically congruent ($\equiv$) (mod p):

$$X^p - X \equiv \prod_{a=0}^{a=p-1}(X+a) \pmod p.$$

We write I_2 thus:

I_3 *The two forms in the two indeterminates* X_0, X_1,

$$D[2,1;p](X_0, X_1) \equiv X_0 X_1^p - X_0^p X_1,$$

$$P[2,1;p](X_0, X_1) \equiv X_0 \cdot \prod_{a_0=0}^{a_0=p-1}(a_0 X_0 + X_1),$$

are identically congruent (mod p):

$$D[2,1;p](X_0, X_1) \equiv P[2,1;p](X_0, X_1) \pmod p.$$

2. We proceed in two steps to a two-fold generalization of Fermat's theorem I_3.

II. *The two forms in the* $k+1$ *indeterminates* $X_0, X_1, \cdots, X_k$,

$$(1)\ D[k+1,1;p](X_0, X_1, \cdots, X_k) \equiv |X_j^{p^i}| \qquad (i,\ j = 0, 1, \cdots, k),$$

$$(2)\ P[k+1,1;p](X_0, X_1, \cdots, X_k) \equiv \prod{}^{*} \sum_g a_g X_g \qquad (g = 0, 1, \cdots, k),$$

—*where the product* $\prod^*$ *embraces the* $(p^{k+1}-1)/(p-1)$ *linear forms* $\sum_{g=0}^{g=k} a_g X_g$ *whose coefficients* a_g $(g=0, 1, \cdots, k)$ *are integers selected from the series* $0, 1, \cdots, p-1$, *in all possible ways, only* *so that for every particular form, first, the coefficients* a_g *are not all* 0, *and, second, of the coefficients* a_g *not* 0 *the one with largest index* g *is* 1 —— *are identically congruent* (mod p):

$$D[k+1, n; p](X_0, X_1, \cdots, X_k) \equiv P[k+1, n; p](X_0, X_1, \cdots, X_k).$$

When we collect into a *class* the totality of integers congruent to one another (mod p), and denote the p incongruent classes by p *marks*, we have in this system of p marks a *field* $F[p]$ *of order* p *and rank 1.* The marks of the field $F[p]$ may be combined by the four fundamental operations of algebra—addition, subtraction, multiplication, division, —the operations being subject to the ordinary abstract operational laws of algebra, the results of these operations being in every case uniquely determined and belonging to the field. *Congruences* ($\equiv$) (mod p) are in the field *equalities* ($=$), and *identical congruencies* ($\equiv$) are *identities* ($\equiv$). The restatement of II in the terminology of the field $F[p]$ is given by setting $n=1$ in its generalization III (§ 3).

3. The second step of generalization of I_3 rests upon *Galois's* generalization of the field $F[p]$ to the *Galois-field* $GF[p^n]$ of *order* p^n, *modulus* p, *and rank* n. This field of p^n marks a is uniquely defined for every $p=$prime, $n=$positive integer. (I have elsewhere proved that every field of finite order s is a Galois-field of order $s=p^n$.)

A form, that is, a rational integral function of certain indeterminates, $X_0, X_1, \cdots, X_k$, is said to *belong* to the $GF[p^n]$ if its coefficients belong to (are marks a of) the $GF[p^n]$. A *linear homogeneous form* $\sum\limits_{g=0}^{g=k} a_g X_g$ belonging to the $GF[p^n]$ is called *primitive* if not all its coefficients a_g are 0, and if of the coefficients a_g not 0 the one with largest index g is 1.

We have then:

III. *The two forms in the* $k+1$ *indeterminates*, $X_0, X_1, \cdots, X_k$,

$$(3)\quad D[k+1, n; p](X_0, X_1, \cdots, X_k) \equiv |X_j^{p^{ni}}| \qquad (i, j=0, 1, \cdots, k),$$

$$(4)\quad P[k+1, n; p](X_0, X_1, \cdots, X_k) \equiv \Pi^* \sum_g a_g X_g \qquad (g=0, 1, \cdots, k),$$

—*where the product* Π^* *embraces the* $(p^{n(k+1)}-1)/(p^n-1)$ *distinct primitive linear homogeneous forms* $\sum\limits_{g=0}^{g=k} a_g X_g$ *belonging to the* $GF[p^n]$ —*are identical:*

$$D[k+1, n; p](X_0, X_1, \cdots, X_k) \equiv P[k+1, n; p](X, X_1, \cdots, X_k).$$

The forms D, P whose identity theorem III affirms have the three characteristic positive integers or *characters* $k+1$, n, p. It is convenient to attach these characters to the notation III for the theorem, and thus to speak of the theorem III$[k+1, n; p]$. This theorem requires proof only for $k \geqq 1$, since for $k=0$, $D \equiv X_0$, $P \equiv X_0$.

4. For the proof of III$[k+1, n; p]$ we need *Galois's* one-fold generalizations of the *Fermat theorems* I:

II$_1'$ *Every mark* a *of the* $GF[p^n]$ *satisfies the equation*

$$a^{p^n} - a = 0.$$

whence,

II$_2'$ *The two forms in the indeterminate* X

$$X^{p^n} - X, \qquad \prod_{a \mid p^n} (X + a)$$

belonging to the $GF[p^n]$ *are identical:*

$$X^{p^n} - X \equiv \prod_{a \mid p^n} (X + a),$$

(where, as always, the subscript-remark $a \mid p^n$ means that the mark a is to run over the p^n marks of the $GF[p^n]$), and further,

II$_3'$ *The two forms in the two indeterminates* X_0 X_1,

$$D[2, n; p](X_0, X_1) \equiv X_0 X_1^{p^n} - X_0^{p^n} X_1,$$
$$P[2, n; p](X_0, X_1) \equiv \prod{}^* (a_0 X_0 + a_1 X_1),$$

— *where the product* $\prod^*$ *embraces the* p^n+1 *distinct primitive linear homogeneous forms* $a_0 X_0 + a_1 X_1$ *belonging to the* $GF[p^n]$ — *are identical:*

$$D[2, n; p](X_0, X_1) \equiv P[2, n; p](X_0, X_1).$$

A (known) corollary to II$_1'$ is also needed. We denote by Λ_k the substitution on the p^n marks a of the $GF[p^n]$ which replaces every mark a by a^{p^k}; obviously $\Lambda_c \Lambda_d = \Lambda_{c+d}$ while by II$_1'$, $\Lambda_n = \Lambda_0 = 1$. We denote by $F_k(X_0, X_1, \cdots, X_k)$ the result obtained by applying the substitution Λ_k to the mark-coefficients of a form $F(X_0, X_1, \cdots, X_k)$ belonging to the $GF[p]$, so that $F_n(X_0, X_1, \cdots, X_k) \equiv F_0(X_0, X_1, \cdots, X_k) \equiv F(X_0, X_1, \cdots, X_k)$. Since in the involution of a multinomial to the p^{th} power the say *intermediate* multinomial coefficients are all divisible by p, we have in the $GF[p^n]$ of modulus p,

$$F_k(X_0, X_1, \cdots, X_k)^p \equiv F_{k+1}(X_0^p, X_1^p, \cdots, X_k^p),$$

whence

II$_1'$ Cor. $\quad F(X_0, X_1, \cdots, X_k)^{p^n} \equiv F(X_0^{p^n}, X_1^{p^n}, \cdots, X_k^{p^n}),$

where $F(X_0, X_1, \cdots, X_k)$ *is any form of the* $GF[p^n]$.

The theorem III$[k+1, n; p]$ bears the same relation to Galois's Fermat's theorem II$'$: II$_3' \equiv$ III$[2, n; p]$: that the theorem II $\equiv$ III$[k+1, 1; p]$ bears to Fermat's theorem I : I$_3 \equiv$ III$[2, 1; p]$. I am communicating then one-fold generalizations II, III of the known theorems I, II$'$; of these II may be looked at as a theorem in the ordinary Gauss-congruence theory, while its generalization III is a theorem in the Galois-field theory.

I give three proofs A, B, C of the general theorem III$[k+1, n; p]$. The proof A depends upon considerations involving the $GF[p^n]$ of rank n alone, and accordingly for $n=1$ this proof A of III$[k+1, 1; p] \equiv$ II may be exhibited in the terminology of the ordinary Gauss-congruence theory. The proofs B and C however depend upon considerations involving the wider $GF[p^{mn}]$ of rank mn $(m \geqq 2k^2)$; they throw a sharper light upon the essential meaning of the theorem III$[k+1, n; p]$ for every n.

Proof A of the theorem III $[k+1, n; p]$. § 5.

Proof by two-based induction. We know that III$[1, n; p]$ and III$[2, n; p] \equiv$ II$_3'$ are true. On the supposition that III$[l-1, n; p]$ and III$[l, n; p]$ $(l > 1)$ are true we prove that III$[l+1, n; p]$ is true.

5. In the determinant $U = | u_{ij} |$ $(i, j = 0, 1 \cdots\cdots, l)$ we denote by U_{ij} the minor complementary to u_{ij}, and have*

$$(5) \qquad (-1)^{l+1} \begin{vmatrix} U_{l1}, & U_{l0} \\ U_{01}, & U_{00} \end{vmatrix} \equiv U . \; | u_{ij} | \qquad \binom{i=1, 2 \cdots, l-1}{j=2, 3, \cdots, l}$$

We set now

$$u_{ij} = X_j^{p^{ni}} \qquad (i, j = 0, 1, \cdots, l),$$

so that we have

$$(6) \quad \begin{aligned} &D[l+1, n; p](X_0, X_1, \cdots, X_l) \equiv + U, \\ &D[l-1, n; p](X_2^{p^n}, X_3^{p^n}, \cdots X_l^{p^n}) \equiv | u_{ij} | \quad \binom{i=1, 2, \cdots, l-1}{j=2, 3, \cdots, l}, \\ &D[l, n; p](X_0, X_2, \cdots, X_l) \equiv (-1)^{l+1} U_{l1}, \\ &D[l, n; p](X_1, X_2, \cdots, X_l) \equiv (-1)^l U_{l0}, \\ &D[l, n; p](X_0^{p^n}, X_2^{p^n}, \cdots, X_l^{p^n}) \equiv - U_{01}, \\ &D[l, n; p](X_1^{p^n}, X_2^{p^n}, \cdots, X_l^{p^n}) \equiv + U_{00}. \end{aligned}$$

Then by substituting the values (6) in the identity (5) and remembering II$_1'$ Cor. and the definition (3) of $D[2, n; p]$ (Y_0, Y_1), we have the fundamental identity

---

* See, for instance, Scott's *Determinants*, p. 57, No. 6.

$$(7)\quad D[2, n; p]\Big(D[l,n;p](X_0,X_2,\cdots,X_l),D[l,n;p](X_1.X_2,\cdots X_l)\Big)$$
$$\equiv D[l+1,n;p](X_0,X_1,\cdots,X_l)\cdot D[l-1,n;p](X_2,X_3,\cdots X_l)^{p^n}.$$

Now, using always the Π^* in the sense defined in the enunciation of III, we have by hypothesis :

$$(8_1)\quad \mathrm{III}[2, n; p]\colon\ D[2, n; p](Y_0, Y_1)\equiv P[2, n; p](Y_0, Y_1)\equiv$$
$$\Pi^*(\beta_0 Y_0+\beta_1 Y_1)\equiv Y_0\cdot\prod_{\beta\,|\,p^n}(\beta Y_0+Y_1),$$

$$(8_2)\quad \mathrm{III}[l-1, n; p]\colon\ D[l-1, n;p](Y_0, Y_1, \cdots, Y_{l-2})\equiv$$
$$P[l-1, n;p](Y_0, Y_1, \cdots, Y_{l-2})\equiv\Pi^*\sum_{h=0}^{h=l-2}\beta_h Y_h,$$

$$(8_3)\quad \mathrm{III}[l, n; p]\colon\ D[l, n;p](Y_0, Y_1, \cdots, Y_{l-1})\equiv$$
$$P[l, n; p](Y_0, Y_1, \cdots, Y_{l-1})\equiv\Pi^*\sum_{h=0}^{h=l-1}\beta_h Y_h.$$

By (8_1) the left side of the identity (7) is

$$(9)\quad D[l, n; p](X_0, X_2, \cdots, X_l)\cdot\prod_{\beta\,|\,p^n}\Big(\beta D[l, n; p](X_0, X_2, \cdots, X_l)$$
$$+D[l, n;p](X_1, X_2, \cdots, X_l)\Big),$$

which, using II_1' and II_1' Cor. in the addition of the determinants, is

$$(10)\qquad D[l, n;p](X_0, X_2, \cdots, X_l)\cdot$$
$$\prod_{\beta\,|\,p^n} D[l, n;p](\beta X_0+X_1, X_2.\cdots, X_l),$$

which by (8_3) is

$$(11)\quad \Pi^*(\delta_0 X_0+\delta_2 X_2+\cdots+\delta_l X_l)\cdot\prod_{\beta\quad p^n}\Pi^*\Big(\gamma_1(\beta X_0+X_1)$$
$$+\gamma_2 X_2+\cdots+\gamma_l X_l\Big).$$

We compare the product (11) with the product (12),

$$(12)\qquad P[l+1, n; p](X_0, X_1, \cdots, X_l)\equiv$$
$$\Pi^*(a_0 X_0+a_1 X_1+a_2 X_2+\cdots+a_l X_l).$$

The factors of (12) are the primitive linear homogeneous forms $\sum_{g=0}^{g=l} a_g X_g$ with no omissions and no repetitions. Every factor of (11) is likewise such a form. And in (11) every such form $\sum_{g=0}^{g=l} a_g X_g$ occurs at least once; any such form $\sum_{g=0}^{g=l} a_g X_g$

with $a_1 + 0$ occurs (since $l > 1$) once and only once, viz., under that $\Pi *$ of the second set in (11) whose forms have $\beta = a_0 / a_1$; any such form $\sum\limits_{g=0}^{g=l} a_g X_g$ with $a_1 = 0$, $a_0 + 0$, occurs once and only once, viz., under the single $\Pi *$ of the first set in (11); any such form $\sum\limits_{g=0}^{g=l} a_g X_g$ with $a_1 = 0$, $a_0 = 0$, occurs in all $1 + p^n$ times, viz., once and of course only once under every $\Pi *$ of (11). Thus the product (11) is:

$$(13) \qquad P[l+1, n; p](X_0, X_1, \cdots, X_l) \cdot \left(\Pi * \sum_{g=2}^{g=l} a_g X_g \right)^{p^n}.$$

When we substitute (13) for the left side of the identity (7) and divide out[*] the second factors, which are identical by the hypothesis (8_2), we have the desired identity

$$(14) \quad \mathrm{III}[l+1, n; p]:$$
$$P[l+1, n; p](X_0, X_1, \cdots, X_l) \equiv D[l+1, n; p](X_0, X_1, \cdots, X_l).$$

Proof B of the theorem III$[k+1, n; p]$. §§ 6–11.

Direct proof of the identity III$\natural[k+1, n; p]$

$$\mathrm{III}\natural. \quad D[k+1, n; p](X_0, X_1, \cdots, X_k) \equiv$$
$$D[k, n; p](X_0, X_1, \cdots, X_{k-1}) \cdot \prod_{\substack{a_r \mid p^n \\ (r=0, \cdots, k-1)}} \left(\sum_{r=0}^{r=k-1} a_r X_r + X_k \right).$$

6. *From the identity* III$\natural[l+1, n; p]$ *for* $l = 1, 2, \cdots, k$, *and in view of the definition* (3) $D[1, n; p](X_0) \equiv X_0$, *we have*

$$(15) \quad D[k+1, n; p](X_0, X_1, \cdots, X_k) \equiv X_0 \cdot \prod_{l=1}^{l=k} \prod_{\substack{a^r \mid p^n \\ (r=0, ,\cdots, l-1)}} \left(\sum_{r=0}^{r=l-1} a_r X_r + X_l \right),$$

viz., *the identity called for by the theorem* III$[k+1, n; p]$.

7. *We introduce the* $GF[p^{mn}]$ *of modulus* p *and rank* mn *where* m *is any integer* $m \geqq 2k^2 \geqq k+1$. The $GF[p^{mn}]$ contains the $GF[p^n]$. The forms entering the theorems III$[k+1, n; p]$, III$\natural[k+1, n; p]$ belong to the $GF[p^n]$, and the theorems then to the $GF[p^n]$-theory, but the forms belong also to the wider $GF[p^{mn}]$, and the proofs B, C of III$\natural$ and III make use of this fact.

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[*] As to the fact that the quotient-forms so obtained are identical, see, for instance, the theorem (3) § 3 of the memoir by **Weber**: *Die allgemeinen Grundlagen der Galoisschen Gleichungstheorie* (*Mathematische Annalen*, vol. 43, pp. 521–549, 1893.)

8. The small Greek letters $\alpha, \beta, \gamma, \cdots$, as heretofore designate marks of the $GF[p^n]$, and the large Greek letters A, B, Γ, $\cdots$, marks of the $GF[p^{mn}]$.

Any t marks A_r $(r=1, 2, \cdots, t)$ of the $GF[p^{mn}]$ determine in the $GF[p^{mn}]$ an *additive-group* $[A_1, A_2, \cdots, A_t \mid GF[p^n]]$ with the *basis-system* $A_1, A_2, \cdots, A_t$ and *field of reference* $GF[p^n]$ containing all possible marks A expressible in the form

$$A = \sum_{r=1}^{r=t} \alpha_r A_r \tag{16}$$

where the $\alpha_r (r=1, 2, \ldots, t)$ are marks of the field of reference $GF[p^n]$.

The p^{tn} marks $A(16)$ are distinct if the particular mark $A = 0$ occurs only once, viz., with the coefficients α_r all 0. In this case, when of course $t \leq m$, the additive-group has the *order* p^{tn} and *rank* t, and the t marks A_r are *linearly independent with respect to** *the* $GF[p^n]$, and obviously any t' of the t marks are likewise linearly independent. If $t < m$, not all the marks of the $GF[p^{mn}]$ are in this additive-group of rank t, and any external mark A_{t+1} forms with the t marks A_r a system of $t + 1$ linearly independent marks, and thus extends the additive-group of rank t to one of rank $t + 1$. We take as a start any mark $A_1 \neq 0$, and by the preceding remarks may affirm: (a) There exist systems of l marks $A_1, A_2, \cdots, A_l$ of the $GF[p^{mn}]$ linearly independent with respect to the $GF[p^n]$ for every l, $l = 1, 2, \cdots, m$; (b) every additive-group $[A_1, A_2 \cdots, A_t \mid GF[p^n]]$ may be exhibited as an additive-group of some rank $l \leq t$ based upon such a system of l linearly independent marks chosen from amongst the original t basal marks A_r.

9. We take now any system of $k + 1$ (not necessarily distinct) marks B_g $(g=0, 1, \cdots, k)$ of the $GF[p^{mn}]$ and consider the two marks

$$\triangle \equiv D[k+1, n;\ p]\ (B_0, B_1, \ldots B_k) \equiv |B_j^{p^{ni}}| \quad (i, j=0, \cdots, 1, k), \tag{17}$$

$$\Pi \equiv P[k+1, n;\ p](B_0, B_1, \cdots, B_k) \equiv \Pi^* \sum_g \alpha_g B_g \quad (g = 0, 1, \cdots, k). \tag{18}$$

Clearly $\Pi = 0$ if and only if the marks B_g are linearly dependent. And further, since from an equation,

$$\sum_{j=0}^{j=k} \alpha_j B_j = 0,$$

---

*In the sequel the $GF[p^n]$ is always the field of reference.

we have, by II_1' Cor., the $k+1$ equations,

$$\left(\sum_{j=0}^{j=k} a_j\, B_j\right)^{p^{ni}} \equiv \sum_{j=0}^{j=k} a_j\, B_j^{p^{ni}} = 0 \qquad (i=0, 1 \ldots, k),$$

we see that if the marks B_g are linearly dependent, then $\triangle = 0$.

(In passing we notice that the remark just made concerning Π and the theorem III$[k+1, n\,;\, p]$ yield the theorem:

A necessary and sufficient condition that $k+1$ *marks* B_g $(g=0, 1, \cdots, k)$ *of a* $GF[p^{mn}]$ *(m any integer) shall be linearly dependent with respect to the* $GF[p^n]$ *is the vanishing of the determinant* $\triangle \equiv | B_j^{p^{ni}} |$ $(i, j=0, 1 \cdots, k)$. (If $m < k+1$, then $\triangle$ vanishes for every system of $k+1$ marks B_g). This yields its still more important corollary:

The rank of the additive-group $[B_0, B_1 \cdots, B_k \mid GF[p^n]]$ *is the same as the rank of the matrix* $(B_j^{p^{ni}})$ $(i, j=0, 1, \cdots, k)$, where a matrix (u_{ij}) $\left(\substack{i=0,\, 1, \ldots,\, k \\ j=0,\, 1, \ldots,\, l}\right)$ is said to have the *rank* r if its every sub-determinant of order $r' > r$ vanishes while at least one sub-determinant of order $r' = r$ does not vanish.

It would not be difficult to establish this theorem and its corollary independently of the theorem III$[k+1, n\,;\, p]$.)

10. Next we consider a system of $k (k+1 \leqq m)$ linearly independent marks A_r $(r=0, 1, \cdots, k-1)$, and select from the p^{kn} marks A of the additive-group $[A_1, A_2 \cdots, A_k \mid GF[p^n]]$ any $k+1$ marks B_g $(g=0, 1, \cdots, k)$. These marks B_g are linearly dependent, for otherwise the additive-group based on them would have the order $p^{(k+1)n}$, whereas it is contained in the additive-group of order p^{kn} based on the k marks A_r. For these marks B_g then (§ 9) $\triangle = 0$, $\Pi = 0$, and so

$$(19)\quad D[k+1, n; p](B_0, B_1, \cdots, B_k) = P[k+1, n; p](B_0, B_1, \cdots, B_k).$$

11. We take now any system of $2\,k^2 (2\,k^2 \leqq m)$ linearly independent marks $E_{rs}\left(\substack{r=0,\, 1, \ldots,\, k-1 \\ s=1,\, 2, \ldots,\, 2k}\right)$ and split it up at once into k systems of $2\,k$ linearly independent (§8) marks.

The system $E_{rs} (s=1, 2, \cdots 2\,k)$ determines the additive-group $[E_{r1}, E_{r2}, \cdots, E_{r2k} \mid GF[p^n]]$ containing besides the mark 0 $p^{2kn}-1$ marks A_r of the form $A_r = \sum_{s=1}^{s=2k} \varepsilon_{rs}\, E_{rs}$, where the ε_{rs} are marks of the $GF[p^n]$ not all 0. We select from each additive-group any mark A_r $(A_r \neq 0)$ $(r=0, 1, \cdots, k-1)$. These k marks A_r are linearly independent, and determine the additive-group $[A_0, A_1, \cdots, A_{k-1} \mid GF[p^n]]$ containing

p^{kn} marks A of the form $A=\sum\limits_{r=0}^{r=k-1} a_r\, A_r$. By §10 we have for every such mark A

$$(20) \qquad D[k+1, n; p](A_0, A_1, \cdots, A_{k-1}, A)=0.$$

Hence the equation for X_k

$$(21) \qquad D[k+1, n; p](A_0, A_1, \cdots, A_{k-1}, X_k)=0.$$

which is of degree p^{kn} in the unknown X_k with the leading coefficient $D[k, n; p](A_0, A_1, \cdots, A_{k-1})$ has as roots the p^{kn} marks A of the form $A=\sum\limits_{r=0}^{r=k-1} a_r\, A_r$. Since the marks A are given equally well in the form $A=-\sum\limits_{r=0}^{r=k-1} a_r\, A_r$, we have the identity in the indeterminate X_k:

$$(22) \qquad D[k+1, n; p](A_0, A_1, \cdots, A_{k-1}, X_k)\equiv$$
$$D[k, n; p](A_0 A_1, \cdots A_{k-1})\cdot \prod_{\substack{a_r \mid p^n \\ (r=0, 1, \dots, k-1)}} \Big(\sum_{r=0}^{r=k-1} a_r\, A_r+X_k\Big).$$

Now consider the two forms in the $k+1$ indeterminates $X_0, X_1, \cdots, X_k$:

$$(23) \qquad D[k+1, n; p](X_0, X_1, \cdots, X_{k-1}, X_k),$$
$$D[k, n; p](X_0, X_1, \cdots, X_{k-1})\cdot \prod_{\substack{a_r \mid p^n \\ (r=0, \dots, k-1)}} \Big(\sum_{r=0}^{r=k-1} a_r\, X_r+X_k\Big).$$

We affirm their identity: this is the theorem III $\natural$ $[k+1, n; p]$. Denoting by $C(X_0, X_1, \cdots, X_{k-1})$, $C'(X_0, X_1, \cdots, X_{k-1})$ the coefficients of any certain same power of X_k in the respective forms (23), we prove the identity in the k indeterminates $X_0, X_1, \cdots, X_{k-1}$:

$$(24) \qquad C(X_0, X_1, \cdots, X_{k-1}) \equiv C'(X_0, X_1, \cdots, X_{k-1}).$$

In the first place these forms are of degree $\leqq p^{kn}$ in each indeterminate. Further by (22)

$$(25) \qquad C(A_0, A_1, \cdots, A_{k-1})= C'(A_0, A_1, \cdots, A_{k-1})$$

where for $r=0,1,\cdots,k-1$ A_r is any of the $p^{2kn}-1$ $(p^{2kn}-1>p^{kn})$ marks $A_r \neq 0$ of the additive-group$[E_{r1}, E_{r2}, \cdots, E_{r2k} \mid GF[p^n]]$. The desired identity (24) follows then by the *identity-theorem* (which is proved in the Galois-field theory just as is the corresponding theorem in the theory of the ordinary general algebraic field):

Identity-theorem. If the two forms of the $GF[p^{mn}]$

$$C(Y_1, Y_2, \cdots, Y_l), \quad C'(Y_1, Y_2, \cdots, Y_l)$$

contain the l indeterminates $Y_1, Y_2, \cdots, Y_l$ to degrees respectively less than the numbers $y_1, y_2, \cdots, y_l$, and if to each indeterminate Y_h ($h=1, 2, \cdots, l$) a certain system of y_h distinct marks A_h may be associated in such a way that the two marks obtained from the two forms by substituting for each indeterminate Y_h any mark A_h of its associated set are equal:

$$C(A_1, A_2, \cdots, A_l) = C'(A_1, A_2, \cdots, A_l):$$

then the two forms are identical:

$$C(Y_1, Y_2, \cdots, Y_l) \equiv C'(Y_1, Y_2, \cdots, Y_l).$$

Proof C of the theorem III $[k+1, n; p]$. §§ 12–14.

Proof by one-based induction. We know that III $[2, n; p] \equiv \text{II}_2'$ is true. On the supposition that III $[l, n; p]$ ($l>1$) is true we prove that III $[l+1, n; p]$ is true.

12. By interchanging two adjacent columns of the determinant $| X_j^{p^{ni}} |$ $(i, j=0, 1, \cdots, l)$ we have:

$$(26) \qquad D[l+1, n; p](\cdots, X_h, X_{h+1}, \cdots) \equiv \\ -D[l+1, n; p](\cdots, X_{h+1}, X_h, \cdots).$$

13. To prove the corresponding property for the product $P[l+1, n; p](X_0, X_1, \cdots, X_l)$ we need the Galois-field generalization of *Wilson's theorem:*

II_2' Cor. 1 $\qquad \prod'_{a \mid p^n} a = -1 \qquad (a + 0),$

whence

II_2' Cor. 2 $\qquad \prod'_{a \mid p^n} a^{p^{hn}} = -1 \qquad (a + 0)$ (*h any integer*),

even for $p=2$, since in a Galois-field of modulus $p=2$ the marks $+1$, -1 are equal.

In view of the definition (4) the two products

$$(27) \qquad P[l+1, n; p](\cdots, X_h, X_{h+1}, \cdots), \\ P[l+1, n; p](\cdots, X_{h+1}, X_h, \cdots)$$

obviously differ only in the factors $\sum_{g=0}^{g=l} a_g X_g$, $\sum_{g=0}^{g=l} \beta_g X_g$ containing both X_h and X_{h+1} and no X_g with $g>h+1$, that is, in the products

$$(28) \qquad \prod'_{\substack{a \mid p^n \\ (a \neq 0)}} \prod_{\substack{a_g \mid p^n \\ (g=0, \ldots, h-1)}} \left(\sum_{g=0}^{g=h-1} a_g X_g + a X_h + X_{h+1} \right),$$

$$\prod'_{\substack{\beta \mid p^n \\ (\beta \neq 0)}} \prod_{\substack{\beta_g \mid p^n \\ (g=0, \ldots, h-1)}} \left(\sum_{g=0}^{g=h-1} \beta_g X_g + \beta X_{h+1} + X_h \right).$$

Setting, for any $\alpha \neq 0, \beta = 1/\alpha, \beta_g = \alpha_g/\alpha$ $(g = 0, 1, \cdots, h-1)$, we find easily that the first product (28) is the second product (28) multiplied by $\prod'_{\alpha \mid p^n} \alpha^{p^{kn}} (\alpha \neq 0)$, *i. e.*, ($\mathrm{II}_2'$ Cor. 2) by -1. Hence indeed:

$$(29) \qquad P[l+1, n; p](\cdots, X_k, X_{k+1}, \cdots) \equiv \\ -P[l+1, n; p](\cdots, X_{k+1}, X_k, \cdots).$$

We have from (26) (29):

$$(30) \qquad D[l+1, n; p](X_0, \cdots, X_{k-1}, X_k, X_{k+1}, \cdots, X_l) \equiv \\ (-1)^{l-k} D[l+1, n; p](X_0, \cdots, X_{k-1}, X_{k+1}, \cdots, X_l, X_k),$$

$$(31) \qquad P[l+1, n; p](X_0, \cdots, X_{k-1}, X_k, X_{k+1}, \cdots, X_l) \equiv \\ (-1)^{l-k} P[l+1, n; p](X_0, \cdots, X_{k-1}, X_{k+1}, \cdots, X_l, X_k).$$

14. Now the two forms

$$(32) \quad D[l+1, n; p](X_0, X_1, \cdots, X_l), \; P[l+1, n, p](X_0, X_1, \cdots, X_l)$$

are of degree p^{ln} in each of the $l+1$ indeterminates X_h $(h = 0, 1, \cdots, l)$; the respective coefficients of $X_l^{p^{nl}}$ are obviously

$$(33) \quad D[l, n; p](X_0, X_1, \cdots, X_{l-1}), \; P[l, n; p](X_0, X_1, \cdots, X_{l-1}),$$ and

hence by (30, 31, 33) the respective coefficients of $X_h^{p^{nl}}$ $(h = 0, 1, \cdots, l)$ are

$$(34) \qquad (-1)^{l-k} D[l, n; p](X_0, \cdots X_{k-1}, X_{k+1}, \cdots, X_l), \\ (-1)^{l-k} P[l, n; p](X_0, \cdots, X_{k-1}, X_{k+1}, \cdots, X_l).$$

From (34) and our hypothesis that III $[k+1, n; p]$ is true for $k + 1 = l$ it follows that the form

$$(35) \quad D[l+1, n; p](X_0, X_1, \cdots, X_l) - P[l+1, n; p](X_0, X_1, \cdots, X_l)$$

is of degree less than p^{ln} in each of the $l+1$ indeterminates X_h $(h = 0, 1, \cdots, l)$. We take in the $GF[p^{mn}]$ $(m \geqq k \geqq l)$ any system of l marks A_r $(r = 0, 1 \cdots, l-1)$ linearly independent with respect to the $GF[p^n]$, and associate with each indeterminate X_h $(h = 0, 1, \cdots, l)$ the p^{ln} marks $A = B_h$ of the additive-group based on the marks A_r. Then, since by § 10 (19) for every such system of $l+1$ (linearly dependent) marks B_h $(h = 0, 1, \cdots, l)$ we have

$$(36) \quad D[l+1, n; p](B_0, B_1, \cdots, B_l) - P[l+1, n; p](B_0, B_1, \cdots B_l) = 0,$$

by the identity-theorem (§ 11) we have the desired identity III$[l+1, n; p]$:

$$D[l+1, n; p](X_0, X_1, \cdots, X_l) - P[l+1, n; p](X_0, X_1, \cdots, X_l) \equiv 0.$$

THE UNIVERSITY OF CHICAGO,
January 30, 1896.

ON THE RUFFINI-ABELIAN THEOREM.

BY PROFESSOR J. PIERPONT.

Gauss having rigorously established in 1799 the fundamental theorem of algebra that every equation of degree n possesses n roots real or imaginary,* it was natural to inquire more closely into the nature of these quantities. When n was less than five, it had long been known that these roots could in every case be expressed as explicit algebraic functions of the coefficients; further it was known that for every degree equations existed of more or less special nature for which this was still true. The unsuccessful attempts of the foremost mathematicians of the century which was just closing to find such expressions for the equation of degree five, when the coefficients were left indeterminate, had rendered it very doubtful if the roots of the general equation of degree greater than four possessed this property. Between the years 1799 and 1813 an Italian mathematician, Ruffini,† made several attempts to establish the justice of these doubts; but his reasoning although highly interesting and valuable is not conclusive, and the question remained open until the publication of Abel's ‡ celebrated argument in 1826, where he proved that it was impossible to express the roots of an equation of degree greater than four, as explicit algebraic functions of the coefficients when these last were left indeterminate.

Abel's demonstration, however, is not all that could be desired. In the first place, as he was ignorant of Ruffini's beautiful researches, the substitution-theoretical part of his paper is unnecessarily roundabout; secondly, in the algebraical part Abel unnecessarily complicates his proof with a classification of functions according to order and degree. Here he commits an error which caused as acute a mind as Hamilton to declare that "it renders it difficult to judge of the validity of his subsequent reasoning." In this, however, Hamilton was misled, the classification in question being entirely superfluous here, however important in other algebraical investigations. In any case it was highly desirable that a theorem of such far-reaching importance and which stands as it were at the very gateway of any algebraical theory of the roots of higher equations, should be demonstrated in a simple, direct and rigorous manner. Such a demonstration was published by Kronecker in the Monatsberichte* of the Berlin Academy for 1879. This paper has not received the attention of writers of treatises on algebra which it deserves; such standard modern treatises† as Serret, Petersen, Carnoy, all contain demonstrations of the theorem which are imperfect. Serret's demonstration is in its algebraical part almost a word for word reproduction of Abel's memoir and thus contains the unfortunate error noted by Hamilton; the demonstration given by Petersen and Carnoy is vitiated by the same tacit postulate made by Ruffini in the Modena treatise of 1813.

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*Gauss, Werke, vol. III. Compare also the interesting paper by Bôcher, Bulletin, May, 1895.

†Cf. Burkhardt. Supplement of the *Zeitschrift für Mathematik u. Physik*, vol. 37.

‡ *Oeuvres Complètes*, 2d edition, vol. I., p. 66. It is interesting to compare this with Abel's first attempt which forms the third paper of the new edition.

Since the publication of Kronecker's paper in 1879, no other proof has appeared, and yet it seems to me that Kronecker in following Abel too closely in the substitution-theoretical part of his demonstration has not given the simplest proof possible; on the other hand, in the algebraical part, in condensing to the utmost the demonstration, he presupposes that his readers are familiar with the form of proof given by Abel. I propose then in the following lines to give a demonstration of the Ruffini-Abelian Theorem which shall be as direct and *self-contained* as possible; in doing this I hope to render a service to a large class of readers whose studies have led them away from algebraical theories, but who would be glad to see this celebrated theorem demonstrated without implying more or less familiarity with the higher parts of algebra. To aid in this I give at the end of this paper § 8 an illustrative example worked out in detail. In addition to this demonstration I give two others: one, a modification of Ruffini's form; the second, Kronecker's modification of Abel's form. As will be observed, the three forms of demonstration have their algebraical part, essentially due to Abel, in common, while the substitution-theoretical part increases in complexity.

§ 1

We begin by giving a few definitions and establishing one or two elementary theorems.

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* p. 205 *Vereinfachung des Abel'schen Beweises*, etc.

† SERRET : *Algèbre supérieure* 5th ed. PETERSEN : *Theorie der algebraischen Gleichungen.* CARNOY : *Cours d'Algèbre supérieure.*

We say $f(\lambda, \mu, \nu\cdots)$ is an integral function of the quantities $\lambda, \mu, \nu, \cdots$ when its calculation requires only the operations of addition, subtraction and multiplication in regard to these quantities; every such function is of the type $\Sigma C\lambda^a\mu^b\nu^c\cdots$ where the coefficients C do not involve $\lambda, \mu, \nu\cdots$ and the exponents $a, b, c\cdots$ are positive integers. The function $f(\lambda, \mu, \nu\cdots)$ is said to be a rational function of $\lambda, \mu, \nu\cdots$ when its calculation requires only the operations of addition, subtraction, multiplication and division in regard to them; every such function is the quotient of two integral functions of $\lambda, \mu, \nu\cdots$ and is thus of the type

$$f(\lambda, \mu, \nu\cdots) = \frac{f_1(\lambda, \mu, \nu\cdots)}{f_2(\lambda, \mu, \nu\cdots)}$$

f_1, f_2, being integral. A function $f(a, b, c, \cdots; \lambda, \mu, \nu\cdots)$ may be an integral function in regard to certain quantities $\lambda, \mu, \nu\cdots$ and rational in regard to certain others, $a, b, c\cdots$; if f involves only the quantities $a, b, c\cdots \lambda, \mu, \nu\cdots$ and certain rational numbers, we say $f(a, b, c\cdots\lambda, \mu, \nu\cdots)$ is an integral function of $\lambda, \mu, \nu\cdots$ whose coefficients are rational functions of $a, b, c\cdots$ with rational-number coefficients, or more shortly f is an integral function of $\lambda, \mu, \nu\cdots$ with coefficients rational in $a, b, c\cdots$.

Let $\varphi(\xi, a, b, c\cdots)$ be an integral function of ξ whose coefficients are rational in $a, b, c\cdots$; if $\xi_1, \xi_2, \xi_3\cdots$ be the values of ξ for which $\varphi=0$,

$$\varphi(\xi) = A(\xi-\xi_1)(\xi-\xi_2)(\xi-\xi_3)\cdots\cdots;$$

similarly if $\eta_1\ \eta_2\ \eta_3\cdots\cdots$ be the zeros of $\psi(\eta, a', b', c'\cdots)$

$$\psi(\eta) = B(\eta-\eta_1)(\eta-\eta_2)(\eta-\eta_3)\cdots\cdots$$

If some of the η's are equal to some of the ξ's as

$$\eta_1=\xi_1,\ \eta_2=\xi_2, \cdots\cdots\ \eta_\mu=\xi_\mu$$

then φ and ψ have a common divisor

$$\delta = (\xi-\xi_1)\cdots(\xi-\xi_\mu)$$

As δ can be obtained by the process of finding the greatest common divisor, and as this process never requires other than rational operations on the coefficients of φ, ψ which are rational in the quantities $a, b, c\cdots a', b', c'$, it follows that δ is an integral function of ξ with coefficients rational in $a, b, c\cdots a,' b,' c'\cdots$. The function $f(\xi, a, b, c\cdots)$ is said to be irreducible in regard to certain quantities $\lambda, \mu, \nu\cdots$ when it is not the product of factors whose coefficients are rational in $\lambda, \mu, \nu, \cdots$; in the same way the equation $f(\xi)=0$ is said to be irreducible under the same circumstances. From this definition it follows that if $f(\xi, a, b, c\cdots)$ be irreducible in regard to $a, b, c\cdots$ and certain quantities $\lambda, \mu, \nu\cdots$ and have a divisor δ in common with $\varphi(\xi, a, b, c\cdots\lambda, \mu, \nu\cdots)$ then φ is divisible by f. For δ as we saw is of the form $\delta(\xi, a, b, c\cdots \lambda, \mu, \nu\cdots)$; which being a factor common to f and φ, requires that f possess a factor whose coefficients are rational in $a, b, c\cdots \lambda, \mu, \nu\cdots$ which is impossible unless $f=\delta$, that is unless φ is divisible by f. Hence:

Theorem I: If

$$f(\xi, a, b, c\cdots) = \xi^m + A_1\xi^{m-1} + \cdots + A_m = 0$$

be irreducible in regard to $a, b, c\cdots \lambda, \mu, \nu\cdots$, and if at the same time ξ satisfy the equation of *less* degree

$$\xi^\mu + B_1\xi^{\mu-1} + \cdots + B_\mu = 0 \qquad \mu < m$$

where the B are rational in $a, b, c\cdots\lambda, \mu, \nu\cdots$ then

$$B_1 = 0 \quad B_2 = 0\cdots\cdots B_\mu = 0$$

We turn now to the substitution theory.
The number of permutations of n things

$$1, 2, 3\cdots\cdots n$$

is $n!$; let

$$l_1 \; l_2 \; l_3\cdots\cdots l_n$$

denote one of these permutations; the operation which replaces 1 by l_1, 2 by $l_2\cdots n$ by l_n is called a substitution and is denoted by the symbol

$$s = \begin{pmatrix} 1 & 2 & 3\cdots n \\ l_1 & l_2 & l_3\cdots l_n \end{pmatrix}$$

where the order of the top row is indifferent if only the l_1 is under 1, l_2 under 2 etc., in each case. As to each of the $n!$ permutations corresponds a substitution, there are $n!$ substitutions affecting the n things $1, 2\cdots n$. I shall speak of them as the substitutions of the symmetric group G or as the substitutions of G. One of these substitutions is

$$\begin{pmatrix} 1 & 2 & 3\cdots n \\ 1 & 2 & 3\cdots n \end{pmatrix}$$

which leaving each element unchanged is called the identical substitution and denoted briefly by 1. Let $s, t, u,\cdots$

be certain substitutions, the operation arising from operating first with s, then with t is denoted by $s.t$ and is called the product of s, t; similarly $s\,t\,u$ is the resulting operation arising from operating first with s, then with t finally with u.

A substitution which replaces each element of the top row by the following, finally the last by the first is called a cycle or a circular substitution; thus

$$\begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 3 & 4 & 1 \end{pmatrix}, \quad \begin{pmatrix} 4 & 2 & 1 & 5 & 3 \\ 2 & 1 & 5 & 3 & 4 \end{pmatrix}$$

are such. Every substitution is the product of cycles, thus

$$\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 \\ 3 & 1 & 7 & 5 & 6 & 4 & 2 \end{pmatrix} = \begin{pmatrix} 1 & 3 & 7 & 2 \\ 3 & 7 & 2 & 1 \end{pmatrix} \begin{pmatrix} 4 & 5 & 6 \\ 5 & 6 & 4 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 \\ 10 & 8 & 3 & 6 & 7 & 5 & 4 & 9 & 2 & 1 & 11 \end{pmatrix} = \begin{pmatrix} 1 & 10 \\ 10 & 1 \end{pmatrix} \begin{pmatrix} 2 & 8 & 9 \\ 8 & 9 & 2 \end{pmatrix} \begin{pmatrix} 3 \\ 3 \end{pmatrix} \begin{pmatrix} 4 & 6 & 5 & 7 \\ 6 & 5 & 7 & 4 \end{pmatrix} \begin{pmatrix} 11 \\ 11 \end{pmatrix}$$

A circular substitution of two elements as (12) is a transposition. To every substitution s corresponds *one* substitution t such that $st=1$; thus if

$$s=\begin{pmatrix} 1 & 2 & 3 & 4 \\ 4 & 3 & 1 & 2 \end{pmatrix}$$

the substitution t is

$$t=\begin{pmatrix} 4 & 3 & 1 & 2 \\ 1 & 2 & 3 & 4 \end{pmatrix}$$

for s converts

$$1\ 2\ 3\ 4$$

into

$$4\ 3\ 1\ 2$$

respectively but each element of this last row is converted by t into

$$1\ 2\ 3\ 4$$

respectively, so the joint effect of s and t, or $s.t$ is to leave each element unchanged $\therefore\ st=1$. The substitution t is called the inverse of s and denoted by s^{-1}, so that $ss^{-1}=1$. For brevity also we put $s.s=s^2$, $s.s.s=s^3$, etc.

Let $x_1\ x_2 \cdots x_n$ be the roots of $x^n+a_1\,x^{n-1}+\cdots+a_n=0$, and let $\varphi\,(x_1, x_2 \cdots x_n;\ \lambda, \mu, \cdots)$ be a rational function of $x_1\,x_2 \cdots x_n$ and certain quantities λ, μ, $\cdots$; if φ remain invariant *in form*, however we permute the indices 1, 2, $3 \cdots n$, or what is the same thing, for all the substitutions of the symmetric group G, φ is said to be a rational symmetric function of $x_1\,x_2 \cdots x_n$.

An elementary theorem of algebra is the following:

THEOREM II. Every rational symmetric function $\varphi(x_1\, x_2, \cdots x_n;\ \lambda, \mu, \nu)$ of the roots $x_1\, x_2 \cdots x_n$ of the equation $x^n + a_1 x^{n-1} + \cdots + a_n = 0$, is a rational function of the coefficients $a_1, a_2 \cdots a_n$ and the quantities $\lambda, \mu, \nu \cdots$.

The function $f(\lambda, \mu, \nu \cdots.)$ is said to be an explicit algebraical function of $\lambda, \mu, \nu \cdots$ when its calculation requires in addition to the rational operations only the extraction of roots in regard to the quantities $\lambda, \mu, \nu \cdots$.

We shall call the equation

$$(a) \qquad f(x) = x^n + a_1 x^{n-1} + \cdots + a_n = 0$$

general when its coefficients are independent variables, and it shall possess an algebraic solution when their exists an explicit algebraic function of the coefficients which satisfies it. We propose to prove that when n is greater than four, no such function exists by showing that the admission of its existence leads to a contradiction.

§ 2

In fact let us seek the most general form of an explicit algebraical function of the coefficients $a_1\, a_2 \cdots a_n$. It may be obtained as Ruffini remarked as follows:

Taking a rational function of the a's as $f_1(a_1\, a_2 \cdots a_n)$ we extract a p_1th root; calling this radical R_1 we form a rational function of the a's and of R_1 as $f_2(a_1 \cdots a_n, R_1)$ and extract a p_2th root; calling this radical R_2 we form a rational function of the a's and of R_1 and R_2 as $f_3(a_1\, a_2 \cdots a_n;\ R_1\, R_2)$, and extract a p_3th root which we call R_3; proceeding in this way every explicit algebraical function x_0 will be defined by the *suite* of equations:

$$(A) \quad \begin{array}{l} R_1^{p_1} = f_1(a_1\, a_2 \cdots a_n),\ R_2^{p_2} = f_2(a_1 \cdots a_n;\ R_1), \cdots \\ R_\lambda^{p_\lambda} = f_\lambda(a_1 a_2 \cdots a_n; R_1 R_2 \cdots R_{\lambda-1}),\ x_0 = f(a_1 a_2 \cdots a_n, R_1, R_2 \cdots R_\lambda) \end{array}$$

the functions f being all rational in regard to the quantities within the parentheses.

We proceed now to show how the *suite* may be simplified without losing its generality. In the first place the exponents p_k may be taken to be primes, since the extraction of an mnth root may be effected by the extraction of an mth, and from that an nth root. Another simplification arises by observing that no loss of generality is incurred by considering f_k to be an integral function of the radicals $R_1\, R_2 \cdots R_{-1k}$ with coefficients rational in $a_1, a_2, \cdots a_n$.

For let for example $f(a_1 \cdots a_n, u)$ be a rational function of u with coefficients rational in $a_1 \cdots a_n$, and

$$(1) \qquad u^p = g\,(a_1 \cdots a_n)$$

then f is of the form

$$(2) \qquad f = \frac{b_0 + b_1 u + b_2 u^2 + \cdots + b_{p-1} u^{p-1}}{c_0 + c_1 u + c_2 u^2 + \cdots + c_{p-1} u^{p-1}} = \frac{\varphi\,(u)}{\psi\,(u)}$$

the coefficients b, c being rational in the a's. It is to be noticed that neither φ nor ψ contains powers of u higher than $p-1$, since all higher powers may be reduced by observing that

$$u^p = g \quad u^{p+1} = u^p . u = g . u, \quad u^{p+2} = u^p . u^2 = g\,u^2 \cdots$$

and g is rational in the a 's.

The expression (2) may be made integral in regard to u as follows: let the p roots of (1) be

$$u, \quad u_1 = \omega\, u \quad u_2 = \omega^2 u \cdots u_{p-1} = \omega^{p-1}\, u$$

where $\omega\ \omega^2 \cdots$ are the roots of the equation

$$(3) \qquad \frac{x^p - 1}{x - 1} = x^{p-1} + x^{p-2} + \cdots + x + 1 = 0$$

The product

$$P = \psi\,(u)\,\psi(u_1) \cdots \psi(u_{p-1})$$

is a symmetric function of the roots of (1) and is thus a rational function $P(a_1 \cdots a_n)$ of $a_1 \cdots a_n$; further the product

$$Q = \psi(u_1)\psi(u_2) \cdots \psi(u_{p-1})$$

considered as a function of ω is symmetric in the roots of (3) and is thus an integral function $Q\,(a_1, \cdots a_n;\ u)$ of u with coefficients rational in $a_1 \cdots a_n$. We have thus, multiplying numerator and denominator by Q,

$$f = \frac{\varphi\ Q}{\psi\ Q} = \frac{\varphi\ Q}{P};$$

and the right hand member is now an integral function of u with coefficients rational in $a_1 \cdots a_n$. Thus every rational function $f\,(a_1 \cdots a_n, u)$ can be expressed as an integral function of u, with coefficients rational in $a_1\, a_2 \cdots a_n$.

Precisely the same reasoning applies to a rational function $f\,(a_1\, a_2 \cdots a_n, u, v)$ where

(4) $$v^q = h\,(a_1 \cdots a_n, u)$$

the function h being an integral function of u with coefficients rational in $a_1 a_2 \cdots a_n$. In fact arranging according to ascending power of v

$$f(u, v) = \frac{l_0 + l_1 v + l_2 v^2 \cdots + l_{q-1} v^{q-1}}{m_0 + m_1 v + m_2 v^2 + \cdots + m_{q-1} v^{q-1}}, = \frac{\lambda\,(v)}{\mu\,(v)}$$

where the coefficients l, m, as rational functions of $a_1 \cdots a_n$, u may be taken as integral functions of u with coefficients rational in $a_1 \cdots a_n$. Making the denominator $\mu(v)$ symmetric in respect to the roots

$$v,\ v_1,\ v_2 \cdots\cdots v_{q-1}$$

of (4) by multiplying numerator and denominator by $\mu\,(v_1) \cdots \mu\,(v_{q-1})$ we see that

$$f\,(u, v) = \frac{\lambda\,(v)\ \mu\,(v_1)\ \cdots\ \mu\,(v_{q-1})}{\mu\,(v)\ \mu\,(v_1)\ \cdots\ \mu\,(v_{q-1})}$$

is an integral function of u, v, with coefficients rational in $a_1 \cdots a_n$. These considerations being applicable to rational functions of any number of radicals, we see that without loss of generality we may assume that the functions f in (A) are integral functions in regard to $R_1\ R_2 \cdots$ with coefficients rational in respect to $a_1\ a_2 \cdots a_n$.

Finally we observe that we can assume that the equations

(5) $$R_k^{p_k} = f_k \qquad k = 1, 2 \cdots \lambda$$

are irreducible in respect to

(6) $$a_1 \cdots a_n;\ R_1, R_2 \cdots R_{k-1}$$

For if (5) be reducible let

(7) $$(X - R_k)(X - R_k') \cdots (X - R_k^{(\mu-1)})$$

be a factor rational in respect to the quantities (6), R_k, $R_k' \cdots R_k^{(\mu-1)}$ being roots of (5); then the absolute term of (7) or

$$R_k\, R_k' \cdots R_k^{(\mu-1)} = \omega\, R_k^{\mu}, \qquad \omega^{p_k} = 1$$

is rational in (6), or if λ satisfy $\lambda\,\mu \equiv 1$ mod p_k, $\omega^{\lambda}\, R_k$ is rational in (6); that is R_k is rational in ω and the quantities (6). But as the determination of ω, as is well known, depends upon the solution of a *suite* of binomial equations whose degrees are the prime factors of $p_k - 1$, the assumption that the equations of the *suite* (A) are irreducible in the above sense, involves no loss of generality.

Thus to sum up: every explicit algebraical function of $a_1\, a_2 \cdots a_n$ is represented by the *suite* (A) where f_k is an integral function of the radicals $R_1, R_2 \cdots R_{k-1}$ whose coefficients are rational in $a_1\, a_2 \cdots a_n$; the equations $R_k^{p_k} = f_k$ are irreducible in respect to $a_1 \cdots a_n$; $R_1 \cdots R_{k-1}$; the exponents p_k are primes, finally

$$\text{(b)} \qquad x_0 = \Sigma A_h R_\lambda^h \qquad h = 0, 1 \cdots p_\lambda - 1$$

where A_h is an integral function of $R_1 \cdots R_{\lambda-1}$, with coefficients rational in $a_1 \cdots a_n$.

§ 3

We impose now the condition that the explicit algebraical function x_0 shall satisfy the equation (a); this will give us a relation between the roots $x_0\, x_1 \cdots$ and the radical R_λ which will indicate a method of simplifying the suite (A) still further. Raising x_0 to the various powers, and substituting these values of $x_0, x_0^2, x_0^3, \cdots x_0^n$ in (a), we have the equation.

$$\text{(1)} \qquad 0 = B_0 + B_1 R_\lambda + B_2 R_\lambda^2 + \cdots + B_{p_\lambda - 1}\; R_\lambda^{p_\lambda - 1}$$

and making use of

$$\text{(2)} \qquad R^{p_\lambda} = f_\lambda$$

to reduce powers of R_λ higher than $p_{\lambda-1}$; the coefficients B are rational in

$$a_0\, a_1 \cdots a_n,\ R_1, R_2 \cdots, R_{\lambda-1}$$

and hence by theorem I., the coefficients B all vanish.* Thus (1) is satisfied when we replace R_λ by $a\, R_\lambda$ where $a^{p_\lambda} = 1$. Hence the expressions obtained from (b) when we replace there R_λ by $a^k R_\lambda$, that is

$$\text{(3)} \qquad x_k = \Sigma A_h\, a^{hk} R_\lambda^h \qquad k = 0, 1 \cdots p_\lambda - 1$$

satisfy (1) and are thus roots of (a). Here all the coefficients A cannot vanish, let A_h be one of these; if we multiply both sides of the p_λ equations (3) respectively by a^{-h}, $a^{-2h} \cdots a^{-(p_\lambda - 1)h}$, and add, we get, remembering that

$$1 + a^r + a^{2r} + \cdots + a^{(p_\lambda - 1)r} = 0$$

the equation

$$\text{(4)} \qquad \Sigma\, a^{-hk}\, x_k = p_\lambda\, A_h R_\lambda^h$$

which gives us the relation in question.

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* We have assumed the *suite* to embrace only irreducible equations; in practice it is convenient to defer the elimination of reducible elements till this stage where we have the means of deciding concerning reducibility.

This relation shows us how we may express x_0 as an integral function of certain radicals

$$P_1, P_2 \cdots\cdots P_\lambda$$

where these radicals enjoy the peculiar property of being integral functions of $x_0\, x_1 \cdots\cdots x_{n-1}$ and certain roots of unity $\alpha, \beta \cdots \omega$ with rational-number coefficients. In fact put†

$$(5) \qquad P_\lambda = p_\lambda A_h R_\lambda^h;$$

this equation together with (2) shows that R_λ is a rational function of P_λ and the quantities $R_1, \cdots R_{\lambda-1}$, since the assumption that (2) is irreducible in respect to $R_1 \cdots R_{\lambda-1}\ P_\lambda$ is, on account of (3) inadmissible. (Theorem I). As R_λ is rational in respect to these quantities so is x_0. That P_λ satisfies a binomial equation of the type

$$P_\lambda^{p_\lambda} = g_\lambda\,(a_1 \cdots a_n, R_1\, R_2 \cdots R_{\lambda-1})$$

is evident; for

$$P_\lambda^{p_\lambda} = p_\lambda^{p_\lambda} A_\lambda^{p_\lambda} R_\lambda^{p_\lambda h} = p_\lambda^{p_\lambda} A_\lambda^{p_\lambda} f_\lambda^h$$

which is a rational function of $a_1 \cdots a_n\ R_1 \cdots R_{\lambda-1}$ which we denote by g_λ.

The radical R_λ being replaced by P_λ, we proceed now to show how we may replace $R_{\lambda-1}$ by $P_{\lambda-1}$. Put

$$\xi_0 = P_\lambda^{p_\lambda} = (\Sigma\alpha^{-hk}\, x_k)^{p_\lambda}$$

and let $\xi_0\, \xi_1\, \xi_2 \cdots$ be the $n\,!$ functions (different or not) arising from applying the $n\,!$ substitutions of the symmetric group to ξ_0. Form now the equation

$$\Phi(\xi) = (\xi - \xi_0)\ (\xi - \xi_1) \cdots\cdots = 0,$$

its coefficients being symmetric in $x_0, x_1, \cdots x_n$ are rational in the coefficients $a_1 \cdots a_n$.

Now a moment's reflection shows that the reasoning applied to the equations

$$f(x) = 0, \qquad x_0 = f_\lambda\,(a_1 \cdots a_n;\ R_1 \cdots R_\lambda)$$

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† This step could obviously be omitted if for $h = 1$, A_h is a rational number.

can be applied word for word to the equations

$$\Phi(\xi)=0, \qquad \xi_0=g_\lambda(a_1\cdots a_n;\ R_1\cdots R_{\lambda-1})$$

In fact referring to the reasoning just made in regard to x_0, R_λ we see that we may write

(b′) $$\xi_0=\Sigma A_r' R^r_{\lambda-1} \qquad r=0,1\cdots p_{\lambda-1}-1.$$

which is the equation analogous to (b).

This value of ξ_0 substituted in $\Phi=0$ gives

(1′) $$0=B_0'+B_1'R_{\lambda-1}+\cdots B'_{p_{\lambda-1}-1}R^{p_{\lambda-1}-1}_{\lambda-1}$$

which is analagous to (1). This shows that the quantities

(3′) $$\xi_s=\sum A_r'\beta^{rs}R^r_{\lambda-1} \qquad s=0,1\cdots p_{\lambda-1}-1, \qquad \beta^{p_{\lambda-1}}=1$$

also satisfy $\Phi=0$. These equations now give as before the relation

(4′) $$\sum \beta^{-rs}x_s=p_{\lambda-1}A_r'R^r_{\lambda-1}$$

analogous to (4). It follows thus, if we put

$$P_{\lambda-1}=p_{\lambda-1}A_r'R^r_{\lambda-1}$$

that $R_{\lambda-1}$ is a rational function of $R_1, R_2,\cdots R_{\lambda-2}$ and $P_{\lambda-1}$ while on the one hand $P_{\lambda-1}$ satisfies a binomial equation of the type

$$P^{p_{\lambda-1}}_{\lambda-1}=g_{\lambda-1}(a_1\cdots a_n;\ R_1, R_2, R_{\lambda-2})$$

and on the other is an integral function of $\xi_0\ \xi_1\cdots$ and β. But $\xi_0\,\xi_1\cdots$ are integral functions of $x_0\ x_1\cdots$ and α, hence $P_{\lambda-1}$ is an integral function of $x_0\,x_1\cdots x_{n-1}$ and the roots of unity α, β; the coefficients of these functions being rational numbers. Returning now to the expression for x_0, namely:

$$x_0=f(a_1\cdots a_n, R_1\cdots R_{\lambda-1}, R_\lambda),$$

we see that we are in the position to replace the radicals $R_{\lambda-1}$, R_λ by the radicals $P_{\lambda-1}$, P_λ. The same reasoning being applicable to the remaining radicals we have the following:

THEOREM III. If an equation is algebraically solvable, we can always give the explicit algebraic expression for a root such a form that all the radicals entering it are integral functions of the roots $x_0\,x_1\cdots x_{n-1}$ and certain roots of unity $\alpha, \beta,\cdots\omega$, the coefficients of these functions being rational numbers;

the radicals themselves are determined by a *suite* of benomial equations

$$(B) \qquad P_1^{p_1} = g_1(a_1\cdots a_n) P_2^{p_2} = g_2(a_1\cdots a_n;\, P_1) \cdots\cdots P_\lambda^{p_\lambda} = g_\lambda\,(a_1\cdots a_n; P_1\, P_2\cdots P_{\lambda-1})$$

while x_0 is given by

$$(C) \qquad x_0 = g\,(a_1\cdots a_n;\, P_1\; P_2\cdots P_\lambda),$$

the functions g being integral functions of $P_1 \cdots\cdots P_\lambda$ with coefficients rational in $a_1\cdots a_n$.

§ 4.

The theorem just proved affords us a sure foundation to construct the substitution-theoretical part of the demonstration; we give first a form due to Ruffini.* Consider the rational integral function of the roots†

$$\varphi\;(x_1\; x_2\cdots x_n)$$

defined by the first equation of (B); let $\varphi_s\; \varphi_{s^2} \cdots\cdots$ be the values of φ after application of $s\; s^2\cdots$ where

$$s = \begin{pmatrix} 1\,2\,3\,4\,5 \\ 2\,3\,4\,5\,1 \end{pmatrix},$$

then since $g_1\;(a_1\cdots a_n)$ is a symmetric function of the roots it remains unchanged for s. Applying then s to the identity

$$\varphi^{p_1} = g_1\;(a_1\cdots a_n)$$

it becomes $\quad \varphi_s^{p_1} = g_1 \quad$ whence

$$(1) \qquad \varphi_s = \beta\,\varphi \qquad \text{where } \beta^{p_1} = 1$$

Applying s to the identity (1) it becomes

$$\varphi_{s^2} = \beta\varphi_s = \beta^2\varphi$$

similarly $\varphi_{s^3} = \beta^3\varphi$, $\varphi_{s^4} = \beta^4\varphi$ and $\varphi_{s^5} = \beta^5\varphi$; but as $s^5 = 1$, $\varphi_{s^5} = \varphi$ and $\beta^5 = 1$.
In the same way, if

$$\sigma = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \end{pmatrix}$$

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*Cf. my paper in *Monatshefte für Mathematik und Physik*, vol. 6, for fuller information concerning this form of proof used by Ruffini in the Modena work already referred to.

† We designate these now by the indices 1, 2...n, as more convenient.

$\varphi_\sigma=\gamma\varphi$ where, as $\sigma^3=1$, $\gamma^3=1$

Hence the application of σ to the identity (1) gives

$$\varphi_{s\sigma}=\beta\varphi_\sigma=\beta\gamma\varphi$$

whence we conclude as above that $(\beta\gamma)^5=1$ since $(s\sigma)^5=1$. Similarly if

$$\rho=\begin{pmatrix} 3 & 4 & 5 \\ 4 & 5 & 3 \end{pmatrix}$$

then $\varphi_\rho=\varphi_{\rho^2}=\varphi$; but since $\rho\sigma=s$, $\varphi_{\rho\sigma}=\varphi_s=\varphi$; hence $\beta=1$ since $\varphi_s=\beta\varphi$. That is φ remains not only unaltered for σ and ρ, but also for s. Consider now the rational function of the roots

$$\psi\,(x_1\,x_2\cdots x_n)$$

defined by the second equation of the *suite* B. The right hand side of the identity

$$\psi^{P_2}=g_2\,(a_1\cdots a_n;\ \varphi)$$

remaining unaltered for the substitutions s, σ, ρ we can reason in regard to ψ precisely as we did concerning φ: hence ψ is unchanged for these substitutions. Proceeding in this way we see that all the radicals $P_1\ P_2\cdots P_\lambda$ remain unaltered for s; but applying s to the identical relation (C) viz:

$$x_1=g\,(\,a_1\cdots a_n;\ P_1\,P_2\cdots P_\lambda\,)$$

we observe that the right hand side remains unaltered whereas the left hand side does not which is an absurdity. The assumption that there exist an explicit algebraical function of the coefficients $a_1\ a_2\cdots a_n$ which satisfies (A), leads thus to a contradiction.

§ 5

An interresting modification of Ruffini's proof is the following: We just saw that P_ν $(\nu=1,\ 2)$ remain unaltered by the circular substitution $\begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \end{pmatrix}$; as the indices 1,2, 3, 4, 5 employed in showing this were any five of the n indices, it follows that P_ν remain unaltered for every circular substitution of three indices. These substitutions we shall call the substitutions of the alternate group which we denote shortly by Γ. We propose to show that P_ν remain unaltered for no other substitutions of the symmetric group G; having established this it follows that the radical P_2 is rational in P_1 which violates the hypothesis made in regard to the *suite* B. In the first place P_1 must change its value for every transposition $\tau=(\lambda, \mu)$ for if it remain unaltered for *any* transposition we show that it remains unaltered for *every* transposition; as every substitution is the product of cycles and as $\begin{pmatrix} 1\,2\,3 \cdots m \\ 2\,3\,4 \cdots 1 \end{pmatrix}$ is the product of transpositions $(1 2)$ $(1 3) \cdots (1\, m)$, P_1 would remain unaltered for every substitution, and then being symmetric functions of the roots, it would be rational in $a_1\, a_2 \cdots a_n$.

Suppose then P_1 remain unaltered for τ, let λ be any substitution of G, then P_1 remains unaltered for $\varphi=\lambda^{-1}\tau\lambda$; for since $P_{\lambda^{-1}\tau}=P_{\lambda^{-1}}$ we have $P_{\lambda^{-1}\tau\lambda}=P_{\lambda^{-1}\lambda}=P_1$

Now, by properly choosing λ, ρ can represent any transposition $(l_\lambda\, l_\mu)$; for let

$$\lambda=\begin{pmatrix} 1 & 2 & 3 \cdots n \\ l_1 & l_2 & l_3 \cdots l_n \end{pmatrix}$$

then λ^{-1} converts

(1) $$l_1\, l_2\, l_3 \cdots l_\lambda \cdots l_\mu \cdots l_n$$

respectively into

$$1\ 2\ 3 \cdots \lambda, \cdots \mu \cdots n$$

which τ converts respectively into

$$1\ 2\ 3 \cdots \mu \cdots \lambda \cdots n$$

which finally λ converts into

(2) $$l_1\, l_2\, l_3 \cdots l_\mu \cdots l_\lambda \cdots l_n\,;$$

that is the substitution $\rho=\lambda^{-1}\,\tau\,\lambda$ converts (1) into (2) so that ρ leaving every index except l_λ, l_μ unaltered and merely transposing these

$$\rho=(l_\lambda,\, l_\mu)$$

Thus every transposition alters P_1; but every pair of transpositions leaves it unchanged, since every such pair is of the type

$$(1\ 2)\ (3\ 4) \text{ or } (1\ 2)\ (2\, 3),$$

but the first being equal to $\begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \end{pmatrix}\begin{pmatrix} 3 & 1 & 4 \\ 1 & 4 & 3 \end{pmatrix}$ and the second equal to $\begin{pmatrix} 1 & 3 & 2 \\ 3 & 2 & 1 \end{pmatrix}$, both are substitutions of Γ. From this we conclude that if any substitution s be the product of m transpositions, it will alter or leave unaltered the radical P_1 according as m is odd or even. This shows that however s be decomposed into a product of transpositions, their number is always odd or always even, and we say s is an odd or even substitution according as m is. As Γ contains all the even substitutions of G, P_1 and hence P_2 also is altered for every substitution not in Γ. The number of substitutions Γ in G is $\frac{1}{2}\, n\,!$; for let

(1) $$s_1=1,\ s_2,\ s_3 \cdots s_m$$

be the even substitution, and

$$t_1\ t_2\ t_3 \cdots t_\mu$$

be the odd; then,

(2) $$t_1,\ s_2 t_1, \cdots\cdots s_m\ t_1$$

are distinct and odd, hence $\mu \geqq m$; also

$$t_1 \cdot t_1,\ \ t_2 t_1,\ \ t_3 t_1, \cdots t_\mu\ t_1$$

are distinct and even, hence $m \geqq \mu\ \therefore\ m=\mu=\frac{1}{2}\ n\,!$ This shows that P_ν acquire but two values for the substitutions of G; for they remain unchanged for (1) while they both change for the substitutions (2) into P_ν', and there are no other substitutions in G.

We now conclude easily that P_2 is rational in P_1. As P_1 is root of a binominal equation, it follows that $P_1' = -P_1$; also the coefficients of

$$\Phi\,(x) = (P_2 + P_2')\ x + P_1\ (P_2 - P_2')$$

being unaltered for the substitutions of G, are rational in $a_1\ a_2 \cdots a_n$; denotingthem by A, B, we have

$$\Phi\,(x) = A\,x + B$$

But $$\Phi\,(P_1) = A\ P_1 + B = 2\ P_1\ P_2,$$

hence $$P_2 = \frac{A\ P_1 + B}{2\ P_1}$$

and P_2 is rational in P_1 and the coefficients $a_1 \cdots a_n$.

§ 6.*

We turn now to Kronecker's form of Abel's proof. From the preceding paragraph we take as proven that P_1 is unaltered for the substitution of Γ, and that every rational function of the roots unaltered for Γ is rational in P_1. Consider the rational function of the roots $\psi\,(x_1 x_2 \cdots x_n) = P_2$ defined by the second equation of the *suite* (B)

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* In anticipation of questions of priority, I remark that this paragraph formed part of a paper I had the honor to read before the Yale Mathematical Club, December 4, 1894. No changes except of the most insignificant nature have been made in the text.

$$\psi^{p_2} = g_2\,(P_1). \tag{1}$$

The $\frac{1}{2}\,n!$ functions (different or not)

$$\psi\ \psi_1\ \psi_2 \cdots\cdots \tag{2}$$

which arise from ψ on applying to it the $\frac{1}{2}\,n!$ substitution of Γ must all be of the form

$$\omega\psi, \qquad \omega^{p_2} = 1$$

since the right hand side of (1) remains unaltered for these substitutions. Thus if ψ remain unaltered for one substitution

$$\sigma = \begin{pmatrix} 1, 2, 3 \cdots n \\ l_1\, l_2\, l_3 \cdots l_n \end{pmatrix}$$

of Γ besides the identical substitution, all the conjugate functions (2) must also remain unaltered for σ. But such a substitution must exist since $\frac{1}{2}\,n! > p_2$; in fact the coefficients of the equation

$$\Phi\,(x) = (x - \psi)\,(x - \psi_1)\,(x - \psi_2) \cdots \tag{3}$$

being symmetric in (2) are unchanged for the substitutions of Γ and are thus rational in P_1.

But $\Phi = 0$ having the root $x = \psi$ in common with the irreducible equation

$$P_2^{p_2} = g_2\,(P_1) \tag{4}$$

is satisfied by all the roots of (4) and hence $\frac{1}{2}\,n\,! \geqq p_2$; as p_2 is prime $\frac{1}{2}\,n\,! > p_2$.

We now show that the conjugate functions (2) remaining unaltered by σ a substitution different from the identical substitution, remain unaltered for every substitution

$s = \begin{pmatrix} 1 & 2 \cdots n \\ h_1 & h_2 \cdots h_n \end{pmatrix}$ of Γ and $P_2 = \psi$ is thus rational in P_1 which is contrary to hypothesis. To do this Kronecker remarks that ψ remains unaltered for every substitution ρ of Γ which changes

$$l_{h_k} \text{ into } h_{l_k} \qquad k = 1, 2 \cdots n$$

In fact $\rho=\sigma^{-1}s^{-1}\sigma s$, for σ^{-1} changes l_{h_k} into h_k which s^{-1} changes into k which is changed by σ into l_k which finally s changes into h_{l_k}. But that $\psi_\rho=\psi$ is manifest since by hypothesis $\psi_\sigma=\psi$ and hence $\psi_{\sigma^{-1}}=\psi$ also, but then $\psi_{\sigma^{-1}s^{-1}}=\psi_{s^{-1}}$ and $\psi_{\sigma^{-1}s^{-1}\sigma}=\psi_{s^{-1}\sigma}=\psi_{s^{-1}}$, hence finally $\psi_{\sigma^{-1}s^{-1}\sigma s}=\psi_{s^{-1}s}=\psi$. Thus if ψ remain unaltered for σ it also remains unaltered for ρ; which enables us to show that ψ is unchanged for at least one circular substitution of three elements. For whatever σ may be it is of one of the five following types:

1° σ contains at least one cycle of more than three letters and is of the form

$$\sigma_1=\begin{pmatrix}1\,2\,3\cdots m\\ 2\,3\,4\cdots 1\end{pmatrix}c_1\,c_2\cdots\cdots$$

where $c_1\,c_2\cdots$ denote other cycles of σ_1

2° it contains two or more cycles of three elements, but no cycle of more elements, and

$$\sigma_2=\begin{pmatrix}1\,2\,3\\ 2\,3\,1\end{pmatrix}\begin{pmatrix}4\,5\,6\\ 5\,6\,4\end{pmatrix}c_1\,c_2\cdots$$

3° it contains only one cycle of three elements and one or more transpositions

$$\sigma_3=\begin{pmatrix}1\,2\,3\\ 2\,3\,1\end{pmatrix}\begin{pmatrix}4\,5\\ 5\,4\end{pmatrix}c_1\,c_2\cdots$$

4° it contains only transpositions, but of these at least three

$$\sigma_4=(1\,2)\;(3\,4)\;(5\,6)\,c_1\,c_2\cdots$$

5° it contains only two transpositions

$$\sigma_5=(1\,2)\;(3\,4)\;(5)\cdots$$

Put now $\rho_\lambda=\sigma_\lambda^{-1}\sigma_\lambda^{-1}\sigma_\lambda s_\lambda$

$$s_1=\begin{pmatrix}1\,2\,3\\ 2\,3\,1\end{pmatrix}\quad s_2=\begin{pmatrix}1\,4\,5\\ 5\,4\,1\end{pmatrix}\quad s_3=\begin{pmatrix}1\,2\,4\\ 2\,4\,1\end{pmatrix}\quad s_4=\begin{pmatrix}1\,3\,5\\ 3\,5\,1\end{pmatrix}=s_5$$

then

$$\rho_1=\begin{pmatrix}1\,2\,4\\ 2\,4\,1\end{pmatrix}\quad \rho_2=\begin{pmatrix}1\,4\,5\,2\,6\\ 4\,5\,2\,6\,1\end{pmatrix}\quad \rho_3=\begin{pmatrix}1\,2\,5\,3\,4\\ 2\,5\,3\,4\,1\end{pmatrix}$$

$$\rho_4=\begin{pmatrix}1\,3\,5\\ 3\,5\,1\end{pmatrix}\begin{pmatrix}2\,6\,4\\ 6\,4\,2\end{pmatrix},\quad \rho_5=\begin{pmatrix}1\,3\,5\,4\,2\\ 3\,5\,4\,2\,1\end{pmatrix}$$

which shows that all the cases may be reduced to 1°. Hence ψ remains unaltered for some circular substitution of three elements

$$\tau = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \end{pmatrix}$$

But if ψ remain unchanged for τ it will also remain invariant for

$$(5) \qquad \begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \end{pmatrix}, \quad \begin{pmatrix} 1 & 2 & 4 \\ 2 & 4 & 1 \end{pmatrix}, \quad \begin{pmatrix} 1 & 2 & 5 \\ 2 & 5 & 1 \end{pmatrix}, \quad \cdots \begin{pmatrix} 1 & 2 & n \\ n & 2 & 1 \end{pmatrix}$$

For let $t = \begin{pmatrix} 1 & 2 & \nu \\ 2 & \nu & 1 \end{pmatrix}$, then ψ being invariant for τ is for $t^{-1} \tau t^{-1} = \omega$ and hence for ω; but $\omega = \begin{pmatrix} 2 & \nu & 3 \\ \nu & 3 & 2 \end{pmatrix}$ and hence

$$\omega^2 = \begin{pmatrix} 2 & 3 & \nu \\ 3 & \nu & 2 \end{pmatrix} \qquad \cdot\ \nu = 4, 5 \cdots n$$

Every substitution of Γ is the product of the substitutions of (5), for every substitution of Γ is the product of a certain number of pairs of transpositions

$$(1\ 2) \quad (1\ 3) \quad \cdots (1\ n).$$

Let $(1\lambda)\ (1\mu)$ be such a pair, then as

$$(1\lambda)\ (1\mu) = \begin{pmatrix} 1 & 2 & \lambda \\ 2 & \lambda & 1 \end{pmatrix} \begin{pmatrix} 1 & 2 & \lambda \\ 2 & \lambda & 1 \end{pmatrix} \begin{pmatrix} 1 & 2 & \mu \\ \mu & 2 & 1 \end{pmatrix}$$

ψ remains unaltered for every substitution of Γ.

This demonstration, while longer than either of the two preceding ones, is interesting from a historical view, as the lemma on the conjugate functions ψ contains as a corollary Cauchy's theorem, which is fundamental in Abel's proof. In fact let

$$\varphi_1 \quad \varphi_2 \quad \cdots \varphi_m$$

be the m distinct values which a rational function φ of $x_1\, x_2 \cdots x_n$ assumes for the substitutions of Γ; as no substitution of Γ leaves *all* the φ*'s* unchanged every substitution of Γ will give rise to a permutation of the above m indices; as there are only $m!$ permutation of m things this number must be certainly as great as $\frac{1}{2} n!$ and

$$m! \overset{=}{>} \tfrac{1}{2} n!$$

Now if $m < n$ then not only is $m! < n!$ but also $m! < \frac{1}{2} n!$ which is impossible. Thus every rational function φ of $x_1 x_2 \cdots x_n$ which takes on more than two values, takes on at least n values; which is Cauchy's theorem.

§ 7.

It being demonstrated that the general equation of degree $n > 4$ possesses no algebraic solution when the coefficients are regarded as independent variables, it lies very near to inquire whether there are equations with constant coefficients, say with rational-number coefficients, which do not possess an algebraic solution. To this very important question the Ruffini-Abelian theorem gives no reply; however in looking over the reasoning employed in the last three paragraphs we observe that its correctness depends upon the obvious fact that every rational equation

$$\Phi(x_1\, x_2 \cdots x_n\, \alpha, \beta \cdots) = 0 \qquad (1)$$

between the roots and certain roots of unity $\alpha, \beta \cdots$, the coefficients being rational numbers, remains true on applying to it any substitution of the alternate group Γ. The question is thus reduced to this: do there exist equations of every degree, whose coefficients are rational numbers such that every relation (1) still subsists on applying any substitution of Γ? That this is so has been shown by Hilbert.* Hence not only has the *general* equation of degree > 4 no algebraic solution, but there exists an infinity of equations of every degree with integral coefficients which possess this property.

§ 8.

We close by giving an illustration of §§ 2, 3 applied to the explicit algebraic function

$$x = \sqrt[3]{-\frac{q}{2} + \sqrt{\frac{q^2}{4} + \frac{p^3}{27}}} + \sqrt[3]{-\frac{q}{2} - \sqrt{\frac{q^2}{4} + \frac{p^3}{27}}}$$

root of the cubic

$$x^3 + px + q = 0 \qquad (1)$$

Here

$$R_1^2 = f_1(p, q) = \frac{q^2}{4} + \frac{p^3}{27}; \quad R_2^3 = f_2(p, q, R_1) = -\frac{q}{2} + R_1$$

$$R_3^3 = f_3(p, q, R_1, R_2) = -\frac{q}{2} + R_1 \qquad (a)$$

$$x_0 = R_2 + R_3 \qquad (2)$$

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* CRELLE, vol. 110, p. 104.

We proceed to replace the *suite* (α) by a *suite* (β) of the type (B) of § 3. Raising (2) to the 2nd and 3rd powers and substituting in (1) we get

$$R_3^3 + 3\,R_2\,R_3^2 + (3\,R_2^2 + p)\,R_3 + (R_2^3 + p\,R_2 + q) = 0$$

or reducing by aid of

$$R_3^3 = f_3 = -\frac{q}{2} - R_1 \qquad R_2^3 = -\frac{q}{2} + R_2$$

we get

$$3\,R_2 . R_3^2 + (3\,R_2^2 + p)\,R_3 + p\,R_2 = 0$$

As the coefficients of this equation are not zero, the last equation of (α) cannot be reducible and R_3 must be rational in the preceding radicals. In fact we know that

$$R_3 = -\frac{p}{3\,R_2}$$

Equation (2) becomes thus

$$(3) \qquad x_0 = R_2 - \frac{{}^1/_3\,p}{R_1 - \frac{1}{2}\,q}\,R_2^2$$

The denominator here is $-R_1 - \frac{1}{2}\,q$ whose conjugate value is $-R_1 - \frac{1}{2}\,q$; hence multiplying numerator and denominator of the second term of the right member of (3) by this last, (3) becomes

$$(4) \qquad x_0 = R_2 - \frac{9}{p^3}\,(R_1 + \tfrac{1}{2}\,q)\,R_2^2$$

which is an integral function of R_1, R_2 with coefficients rational in p, q. The value of x_o given by (4) substituted in (1) gives

$$0 = B_o + B_1\,R_2 + B_2\,R_2^2,$$

where

$$B_0 = R_1 - \frac{q}{2} - \frac{3^6}{p^6} . \frac{p^6}{3^6}\,(R_1 + \tfrac{1}{2}\,q)$$

$$B_1 = -\frac{3^3}{p^2}\,\frac{p^3}{3^3} + p$$

$$B_2 = \frac{3^5}{p^4}\,\frac{p^3}{3^3}\,({}_1 + \tfrac{1}{2}\,q) - \frac{3^2}{p}\left(R_1 + \frac{q}{2}\right)$$

which are all identically equal to zero. Hence not only is (4) a root of (1) but so also are the quantities which we deduce from it in replacing R_2 by ωR_2 and $\omega^2 R_2$ $(\omega^3 = 1)$. Denoting these roots respectively by $x_0\, x_1\, x_2$ we have

$$x_0 = R_2 - \frac{3^2}{p^2}(R_1 + \tfrac{1}{2} q) R_2^2$$

$$x_1 = \omega R_2 - \frac{3^3}{p^2}(R_1 + \tfrac{1}{2} q)\omega^2 R_2^2$$

$$x_2 = \omega^2 R_2 - \frac{3^2}{p^2}(R_1 + \tfrac{1}{2} q)\omega R_2^2$$

As the coefficicnt of R_2 is here 1 we do not need to replace R_2 by a new radical. Solving we get

$$R_2 = \tfrac{1}{3} \sum \omega^{-k} x_k \qquad k = 0, 1, 2$$
$$= \frac{x_0 + \omega^2 x_1 + \omega x_2}{3}$$

The suite (α) is now replaced by

$$(\beta) \qquad \begin{aligned} R_1^2 &= f_1 \quad R_2^3 = f_2 \\ x_0 &= g(p, q, R_1, R_2,) \\ &= R_2 - \frac{9}{p^2}(R_1 + \tfrac{1}{2} q) \mathrm{R}_2^2 \end{aligned}$$

and the first step has been accomplished.

We have now to reason in the same way in regard to the remaining radical R_1. To this end we form the equation $\varphi(\xi) = 0$ whose roots are the values of

$$\xi_0 = R_2^3 = \left(\frac{x_0 + \omega^2 x_1 + \omega x_2}{3}\right)^3$$

As ξ_0 acquires only two distinct values, Φ will be the cube of an integral function of ξ with coefficients rational in p, q; this is easily found to be

$$\varphi(\xi) = \xi^2 + q\,\xi - \tfrac{1}{27} p^3,$$

Substituting here the value of ξ given in (β), $\varphi(\xi) = 0$ becomes

$$(R_1 - \tfrac{q}{2})^2 + q\,(R_1 - \tfrac{q}{2}) - \tfrac{1}{27} p^3 = 0,$$

or

$$B_0 + B_1 R_1 = 0$$

where

$$B_0 = \tfrac{1}{4} q^2 + \tfrac{1}{4} q^2 - \tfrac{1}{2} q^2 + \tfrac{1}{27} p^3 - \tfrac{1}{27} p^3$$
$$B_1 = q - q$$

vanish identically; whence not only

$$\xi_0 = R_1 - \frac{q}{2}$$

but also

$$\xi_1 = -R_1 - \frac{q}{2}$$

is a root of $\varphi = 0$.
Whence

$$R_1 = \tfrac{1}{2}(\xi_0 - \xi_1) = \frac{1}{2\cdot 3^3}\left[(x_0 + \omega^2 x_1 + \omega^2 x_2)^3 - (x_0 + \omega x_1 + \omega^2 x_2)^3\right]$$

$$= -\frac{\sqrt{-3}\,\sqrt{\Delta}}{2.9},$$

where $\Delta = (x_0 - x_1)^2 (x_1 - x_2)^2 (x_1 - x_2)^2$ is the discriminant of (1). Thus the *suite* (β) has the character (B).

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ON CERTAIN SUB-GROUPS OF THE GENERAL PROJECTIVE GROUP.

BY PROFESSOR HENRY TABER.

[Read at the January meeting of the Society, 1896.]

§ 1

In what follows a linear transformation homogeneous in n variables as

$$\begin{aligned} x_1' &= a_{11}\, x_1 + a_{12}\, x_2 + \cdots + a_{1n}\, x_n, \\ x_2' &= a_{21}\, x_1 + a_{22}\, x_2 + \cdots + a_{2n}\, x_n, \\ &\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ x_n' &= a_{n1}\, x_1 + a_{n2}\, x_2 + \cdots + a_{nn}\, x_n, \end{aligned}$$

will be denoted by the single letter A. If x_1, x_2, etc., are the Cartesian coördinates of a point in n-fold space, the transformation A is a homogeneous strain; and the totality of transformations A constitutes the group of homogeneous strains in n-fold space. If we consider only transformations A of non-zero determinant, we obtain Lie's *general linear homogeneous group*. The group of transformations A of determinant $+1$ is termed by Lie the *special linear homogeneous group*.

If B denotes the transformation

$$x_r'' = b_{r1} x_1' + b_{r2} x_2' + \cdots + b_{rn} x_n' \; (r=1, 2, \cdots n),$$

the transformation resulting from the successive application to n-fold space of A and B (in the order named), defined by the system of equations

$$x_r''=\sum_1^n{}_i\, b_{ri}\, a_{i1} \cdot x_1 + \sum_1^n{}_i\, b_{ri}\, a_{i2} \cdot x_2 + \cdots + \sum_1^n{}_i\, b_{ri}\, a_{in} \cdot x_n$$

$$(r=1, 2, \cdots n),$$

will be denoted by $B\,A$.

If the transformation B is identical with A, this may be indicated by writing $B=A$; and in this case the transformation $BA=BB$ may be denoted by B^2. Similarly the transformation resulting from the repetition $m-1$ times of the transformation B (that is, the transformation resulting from m successive applications of B) may be denoted by B^m, which may be termed the m^{th} *power* of B.

If for $r, s = 1, 2, \cdots n$,

$$b_{rr} = 1, \; b_{rs} = 0 \; (r \neq s),$$

B is the *identical transformation*. The letter I will be employed to denote the identical transformation. We have $IA = A\,I = A$. We may denote the transformation for which

$$b_{rr} = \rho, \; b_{rs} = 0 \; (r \neq s),$$

by ρI; and the transformation $\rho I. A$, defined by the equations $x_r'=\rho\, a_{r1}\, x_1 + \rho\, a_{r2}\, x_2 + \cdots + \rho\, a_{rn}\, x_n$ $(r = 1, 2, \ldots n)$ may be denoted simply by $\rho\, A$. In particular, the transformation defined by the equations

$x_r'=-\, a_{r1}\, x_1 - a_{r2}\, x_2 - \cdots - a_{rn}\, x_n$ $(r = 1, 2, \ldots n)$ will be denoted by $-A$.

The transformation B is *infinitesimal* if

$$b_{rr} = 1+c_{rr}\,\delta t, \quad b_{rs}=c_{rs}\,\delta t \quad (r \neq s),$$

where c_{rr} and c_{rs} are constants, and δt is an infinitesimal quantity.

Following Sylvester, I shall say that the nullity of B is m if the minor determinants of order $n-m+1$ of the matrix

$$\begin{Vmatrix} b_{11}, & b_{12}, & \cdots\cdots & b_{1n} \\ b_{21}, & b_{22}, & \cdots\cdots & b_{2n} \\ \cdots & \cdots & \cdots & \cdots \\ b_{n1}, & b_{n2}, & \cdots\cdots & b_{nn} \end{Vmatrix}$$

are all zero, but not all the minors of order *n-m*.

If we denote by A_λ the transformation

$$\begin{aligned} x_1' &= (a_{11}-\lambda)\, x_1 + a_{12}\, x_2 + \cdots a_{1n}\, x_n, \\ x_2' &= a_{21}\, x_1 + (a_{22}-\lambda)\, x_2 + \cdots + a_{2n}\, x_n, \\ &\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ x_n' &= a_{n1}\, x_1 + a_{n2}\, x_2 + \cdots + (a_{nn}-\lambda)\, x_n, \end{aligned}$$

where λ is a root of multiplicity μ of the *characteristic equation* of A,

$$\begin{vmatrix} a_{11}-z, & a_{12}, & \cdots\cdots & a_{1n} \\ a_{12}, & a_{22}-z & \cdots\cdots & a_{2n} \\ \cdots & \cdots & \cdots & \cdots \\ a_{n1}, & a_{n2}, & \cdots\cdots & a_{nn}-z \end{vmatrix} = 0,$$

the nullity of A_λ is at least one, and cannot exceed μ. If the nullity of A_λ is equal to μ, the nullity of A_λ^2, A_λ^3, etc., is also μ. On the other hand, if the nullity of A_λ is less than μ, the nullity of A_λ^2 is greater by one, at least, than the nullity of A_λ; and the nullity of successive powers of A_λ increases until a power of exponent ν is attained whose nullity is μ. The nullity of the $(\nu+1)^{th}$ and higher powers of A_λ is then also μ. If we denote by

$$\mu_1, \mu_2, \cdots \mu_{\nu-1}, \mu_\nu = \mu,$$

respectively, the nullities of

$$A_\lambda, A_\lambda^2, \cdots A_{\lambda,}^{\nu-1} A_\lambda^\nu,$$

we have

$$\mu_1 \geqq \mu_2 - \mu_1 \geqq \mu_3 - \mu_2 \geqq \ldots \geqq \mu_\nu - \mu_{\nu-1} \geqq 1.$$

I shall term the numbers $\mu_1, \mu_2 \cdots \mu_\nu$, the numbers *belonging* to the root λ of the characteristic equation A. We can always find a transformation A which shall have any complex number λ as a root of its characteristic equation, and any set of numbers, subject only to the above conditions, as the numbers belonging to that root. To subject the numbers belonging to any root of the characteristic equation of A to any condition other than those given above, is of course, equivalent to imposing certain conditions upon the coefficients of A.

The real linear homogeneous group G, for n variables, and the group $\mathfrak{G}$ of real projective transformations in $(n-1)$-fold space are connected as follows. If the variables x_1, x_2, etc., are regarded as the homogeneous coördinates of a point in $(n-1)$-fold space, the real linear transformation A homogeneous in the x's determines a corresponding real projective transformation $\mathfrak{A}$ in $(n-1)$-fold space. In the determination of this projective transformation, we are concerned only with the ratios of the coefficients of A; consequently, the transformations A and ρA of group G (where ρ is any real number) correspond to the same projective transformation $\mathfrak{A}$. The identical transformation I and the transformation ρI of group G correspond to the same projective transformation $\mathfrak{I}$, the identical transformation of group $\mathfrak{G}$. Let B be a transformation of group G corresponding to the real projective transformation $\mathfrak{B}$: if $A = B^m$, then $\mathfrak{A} = \mathfrak{B}^m$; and, if $\mathfrak{A} = \mathfrak{B}^m$, then $\rho A = B^m$ for some real quantity ρ. Consequently, if either A or ρA is the m^{th} power of a transformation of group G, the corresponding projective transformation $\mathfrak{A}$ of group $\mathfrak{G}$ is the m^{th} power of some transformation of this group, and not otherwise. Further, if B is infinitesimal, the corresponding projective transformation $\mathfrak{B}$ differs infinitely little from the identical transformation of group $\mathfrak{G}$. Therefore, if either A or ρA can be generated by the repetition of an infinitesimal transformation of group G, the corresponding projective transformation of group $\mathfrak{G}$ can be generated by the repetition of an infinitesimal transformation of this group and not otherwise.

If $\bar{G}$ denotes the group of real linear transformations of determinant $+1$ homogeneous in n variables (the real special linear homogeneous group), every transformation $\bar{A}$ of group $\bar{G}$ determines a (corresponding) transformation of group $\mathfrak{G}$. If n the number of variables is odd, there is a one to one correspondence between the transformations of group $\bar{G}$ and the transformations of the real projective group. For in this case (that is, if n is odd), if A is a real linear transformation homogeneous in the variables corresponding to the real projective transformation $\mathfrak{A}$, the transformation $\bar{A} = \frac{1}{a} A$, where a denotes the real n^{th} root of the determinant of A, is a transformation of group $\bar{G}$ and the only transformation of this group the ratios of whose coefficients are the same as the ratios of the coefficients of

A. Consequently, when n is odd, if $\overline{A}$ is the m^{th} power of a transformation of group G, the corresponding real projective transformation $\mathfrak{A}$ is the m^{th} power of a transformation of group $\mathfrak{G}$ and conversely; also if $\overline{A}$ can be generated by the repetition of an infinitesimal transformation of group G, $\mathfrak{A}$ can be generated by the repetition of an infinitesimal transformation of group $\mathfrak{G}$, and conversely. On the other hand, if n is even, and if the determinant of A (the real linear transformation homogeneous in the variables, determining the real projective transformation $\mathfrak{A}$) is positive, there are corresponding to $\mathfrak{A}$ two transformations of group $\overline{G}$ obtained by dividing each coefficient of A by either the positive or by the negative real n^{th} root of the determinant of A; but if the determinant of A is negative there is no transformation of group $\overline{G}$ corresponding to $\mathfrak{A}$.

§ 2.*

In the *Comptes Rendus*, volume 94, Sylvester showed that every transformation A of non-zero determinant is the m^{th} power for any positive integer m of some transformation B. That is, if A is any transformation of non-zero determinant, a transformation B can always be found, for any positive integer m, such that $A = B^m$. In his paper, Sylvester also gave a formula expressing the coefficients of B as algebraic functions of the coefficients of A. From a subsequent theorem of Sylvester's,† it follows that, if m is sufficiently great, we may take B to differ as little as we please from the identical transformation.

For real linear transformations homogeneous in n variables we have the following theorems.

Theorem I. *If the transformation A is real and of non-zero determinant (or of determinant $+ 1$), a real transformation B of non-zero determinant (or of determinant $+ 1$) can always be found of which A is the $(2m + 1)^{th}$ power for any odd number $2m + 1$. But in order that A may be any even power of a real transformation B of non-zero determinant (or of determinant $+1$), it is necessary and sufficient that all the numbers belonging to each negative root of the characteristic equation of A shall be even. Therefore, in order that the real transformation A of non-zero determinant (or of determinant $+ 1$) may be any even power of a real transformation B of non-zero determinant (or of determinant*

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* Throughout this section, by transformation is meant a linear transformation homogeneous in n variables.

† *Johns Hopkins University Circulars.*

$+1$) *it is necessary and sufficient that A shall be the second power of a real transformation of non-zero determinant* (*or of determinant* $+1$).

Theorem II. *If A is the second power of a real transformation of non-zero determinant* (*or of determinant* $+1$), *a real transformation B of non-zero determinant* (*or of determinant* $+1$) *can always be found which shall be as nearly as we please equal to the identical transformation, and such that for some positive integer m sufficiently great B^m shall be equal to A. Consequently, any transformation that is the second power of a real transformation of non-zero determinant can be generated by the repetition of a real infinitesimal transformation; and, in particular, a transformation that is the second power of a real transformation of determinant* $+1$ *can be generated by the repetition of an infinitesimal transformation of determinant* $+1$.

Theorem III. *If the real transformation A of non-zero determinant* (*or of determinant* $+1$) *is not the second power of a real transformation of non-zero determinant* (*or of determinant* $+1$), *nevertheless, a real transformation B of non-zero determinant* (*or of determinant* $+1$) *can be found whose coefficients are algebraic functions of a parameter t such that A shall be the limit towards which the transformation B^2 approaches as t approaches zero, provided the multiplicity of each negative root of the characteristic equation of A is even.*

If λ is any root of the characteristic equation of the real transformation B, and if, for $r, s=1, 2, \cdots n$,

$$|1-b_{rr}| < \epsilon, \quad |b_{rs}| < \epsilon \ (r + s),$$

where ϵ is any positive quantity, then provided ϵ is sufficiently small, the absolute value of $1-\lambda$ may be made as small as we please; and consequently no root of the characteristic equation of B will be negative or equal to zero. Therefore, by *Theorem* I, if ϵ is sufficiently small, B is the second power of a real transformation B_1 of non-zero determinant; and, in particular, if B is of determinant $+1$, B_1 may be so chosen that its determinant shall be equal to $+1$. For any positive integer m, B^m is equivalent to the second power of $B_1{}^m$; that is, $B^m=(B_1^2)^m=(B_1{}^m)^2$. Whence we have the following theorem.

Theorem IV. *The repetition of a real infinitesimal transformation results in a transformation which is the second power of a real transformation. In particular, the repetition of a real infinitesimal transformation of determinant* $+1$ *results in a transforma-*

tion which is the second power of a real transformation of determinant $+1$.

From *Theorem* III follows:

Theorem V. *By the repetition of a real infinitesimal transformation, we may approach as near as we please to any real transformation A of non-zero determinant, provided the multiplicity of each negative root of the characteristic equation of A is even. In particular if these conditions are satisfied, and if the determinant of A is $+1$, we may approach as near as we please to A by the repetition of a real infinitesimal transformation of determinant* $+1$.

If the transformation A is not the second power of a real transformation, it may be that $-A$ is the second power of a real transformation. The roots of the characteristic equation of $-A$ are the negatives of the roots of the characteristic equation of A. Whence follows from *Theorem* I:

Theorem VI. *In order that the real transformation $-A$ of non-zero determinant shall be the second power of a real transformation, it is necessary and sufficient that the numbers belonging to each positive root of the characteristic equation of A shall all be even.*

§ 3.

In the *Bulletin* for July, 1894, I pointed out that not every orthogonal substitution can be generated by the repetition of an infinitesimal orthogonal substitution, and if G is the group of proper orthogonal substitutions I showed that

Theorem VII. *If we designate a transformation of the group G as of the first or second kind according as it is or is not the second power of a transformation of the group, then every transformation of the first kind is the m^{th} power of a transformation of the group for any positive integer m, and can be generated by the repetition of an infinitesimal transformation of the group. By definition no transformation of the second kind is any even power of a transformation of the group. Nor can any transformation of the second kind be generated by the repetition of an infinitesimal transformation of the group. But every transformation of the second kind is the $(2m+1)^{th}$ power of a transformation of the group for any odd exponent* $2m+1$.

In a note at the end of this paper, I mentioned that *Theorem* VII was true of the group of linear transformations homogeneous in n variables that transform into itself a bilinear form (of non-zero determinant understood) with cogrediant variables. This was afterwards demonstrated for an alternate bilinear form in a paper published in volume 46 of the *Mathematische Annalen.*

In a paper presented to the *American Academy of Arts and Sciences*, April, 1895, I have shown that *Theorem* VII holds for the *class* of linear transformations homogeneous in n variables that transform a bilinear form (of non-zero determinant) with contragrediant variables into itself. The transformations of this class that are commutative form a a group for which also *Theorem* VII holds.

My attention has recently been called to a note in the *Bulletin* for November, 1893,* stating that Professors Engel and Study have found that not every transformation of the special linear homogeneous group for two variables can be generated by the repetition of an infinitesimal transformation of that group. There is nothing in the note to show that the *real* special linear homogeneous group is referred to.†

I learn from Prof. H. B. Newson that he has found that not every transformation of the real projective group can be generated by the repetition of an infinitesimal transformation of that group. This discovery suggested the extension of *Theorem* VII to other sub-groups of the projective group. It follows at once from Newson's discovery that not every transformation of the real linear homogeneous group can be generated by the repetition of an infinitesimal transformation of this group; and further that not every transformation of the real special linear homogeneous group for an odd number of variables can be generated by the repetition of an infinitesimal transformation of this group. For could every transformation of either of these groups be generated by the repetition of an infinitesimal transformation of that group, then, as pointed out in § 1, every transformation of the real projective group could be generated by the repetition of an infinitesimal transformation of this group, which is not the case.

It follows from *Theorems* I, II and IV, that *Theorem* VII is true for the real linear homogeneous group for n variables, and also for the group of real linear transformations of determinant +1 homogeneous in n variables (the real special linear homogeneous group); and, from what is stated in the latter part of § 1, that *Theorem* VII holds for the real projective group.

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* Page 66.

† *Theorem* VII holds for the special linear homogeneous group in two variables; and the transformation $x' = -x + y$, $y' = -y$ is not the second power of a transformation of this group, and therefore cannot be generated by the repetition of an infinitesimal transformation of this group. From the note it would appear that the identical transformation $x' = x$, $y' = y$, and the involutional transformation $x' = -x$, $y' = -y$ can not be generated by the repetition of an infinitesimal transformation of the group, which is erroneous.

In volume 26, page 364 of the *Proceedings of the London Mathematical Society*, I have shown that in order that an orthogonal substitution may be of the first kind (that is, in order that a transformation of the orthogonal group may be generated by the repetition of an infinitesimal orthogonal transformation), it is necessary and sufficient that either -1 shall not be a root of the characteristic equation of the substitution or that the numbers belonging to the root -1 shall all be even. In a paper read at the Summer Meeting, 1895, of the *American Mathematical Society* (to appear in the *Quarterly Journal of Mathematics*) I showed that a transformation of the group of linear transformations homogeneous in the variables that transform an alternate bilinear form with cogrediant variables into itself will be of the first kind under the same conditions.*

By *Theorem* I, in order that a transformation of the real linear homogeneous group (or of the real special linear homogeneous group) shall be of the first kind it is necessary and sufficient that the numbers belonging to each negative root of the characteristic equation of the transformation shall all be even.

Therefore, by § 1, in order that a transformation of the real projective group shall be of the first kind, it is necessary and sufficient, either that the numbers belonging to each positive root of the characteristic equation of the transformation shall all be even, or else that the numbers belonging to each negative root of the characteristic equation shall all be even; provided there are both positive and negative real roots. Otherwise no condition is necessary.

In the article in the *Bulletin* on orthogonal substitution above mentioned I showed that by the repetition of an infinitesimal orthogonal substitution we can approximate as nearly as please to any proper orthogonal substitution of the second kind. In the above mentioned paper in the *Mathematische Annalen* I showed that by the repetition of an infinitesimal transformation of the group of linear transformations homogeneous in the variables that transform an alternate bilinear form into itself we can approximate as nearly as we please to any transformation of the second kind belonging to this group.

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* In the paper I also showed that every orthogonal transformation of this group is of the first kind. Also, if $\mathfrak{F}$ denotes a bilinear form of non-zero determinant with cogrediant variables, and $\mathfrak{F}'$ its conjugate; and if neither the determinant of $\mathfrak{F} + \mathfrak{F}'$ nor of $\mathfrak{F} - \mathfrak{F}'$ is zero, that then every linear transformation homogeneous in the variables that transforms $\mathfrak{F}$ into itself is the m^{th} power of such a transformation for any exponent m.

From *Theorem* V it follows that by the repetition of an infinitesimal transformation of the real linear homogeneous group (or of the real special linear homogeneous group) we may approximate as nearly as we please to any transformation of the second kind belonging to this group, provided the multiplicity of each negative root of the characteristic equation of this transformation is even.

Therefore by the repetition of an infinitesimal transformation of the real projective group we may approximate as nearly as we please to any transformation of the second kind belonging to this group, provided that either the multiplicity of each positive root of the characteristic equation of this transformation is even or else that the multiplicity of each negative root of the characteristic equation is even.

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POSTSCRIPT.

The general real linear group and the special real linear group. The totality of linear transformations

$$y_r' = a_{r1} y_1 + a_{r2} y_2 + \cdots + a_{r,\, n-1} y_{n-1} + a_{r,\, n} \quad (r = 1, 2, \cdots \overline{n-1})$$

for which the determinant $\sum \pm a_{11} a_{22} \cdots a_{n-1,\, n-1}$ is not zero form a group termed by Lie the *general linear group*. The sub-group of transformations for which the determinant $\sum \pm a_{11} a_{22} \cdots a_{n-1,\, n-1}$ is $+1$, Lie terms the *special linear group*. These two groups are obviously identical with the two groups of linear transformations A that are respectively of non-zero determinant and of determinant $+1$, homogeneous in n variables and defined by the equations

$$x_r' = a_{r1} x_1 + a_{r2} x_2 + \cdots + a_{rn} x_n \quad (r = 1, 2, \cdots \overline{n-1}),$$
$$x_n' = x_n,$$

For the general real linear group, and for the special real linear group, *Theorem* VII holds.* The conditions necessary and sufficient that a transformation A of the general real linear group (or of the special real linear group) may be of the first kind are that the numbers belonging to each negative root of the characteristic equation of A shall all be even. By the repetition of an infinitesimal transformation of the real linear group we may approximate as nearly as we please to any transformation A of the second kind belonging to this group, provided the multiplicity of each negative root of the characteristic equation of is A even, but not otherwise. In particular, if these conditions are satisfied (but not otherwise), and if A is of determinant $+1$, we may approximate as nearly as we please to A by the repetition of an infinitesimal transformation of the special real linear group.

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* Every transformation of the general linear group is the m^{th} power of a transformation of this group for any exponent m, and can be generated by the repetition of an infinitesimal transformation of this group, if we consider transformation of this group both imaginary and real. The same is true of the special linear group.

Special linear homogeneous group. Let A be any linear transformation (real or imaginary) of determinant $+1$ homogeneous in n variables, that is, let A be any transformation of the special linear homogeneous group. Let

$$\lambda' = r_1 e^{\theta_1 \sqrt{-1}}, \qquad \lambda'' = r_2 e^{\theta_2 \sqrt{-1}}, \text{ etc.},$$

(where θ_1, θ_2, etc., are the smallest positive arguments of λ', λ'', etc.,) be the roots of the characteristic equation of A of respectively multiplicity m', m'', etc. Since the determinant of A is $+1$, we have

$$r_1^{m'} \, r_2^{m''} \cdots\cdots = 1,$$

$$m' \, \theta_1 + m'' \, \theta_2 + \cdots\cdots = 2 \, k \, \pi,$$

for some positive integer k.

If $k=0$, the transformation A can be generated by the repetition of an infinitesimal transformation of the special linear homogeneous group. The conditions in general necessary and sufficient in order that A may be generated by the repetition of an infinitesimal transformation of this group are somewhat complicated; they depend upon k, and upon the numbers belonging to the roots λ of the characteristic equation of A. The conditions may be exhibited as follows.

Let the numbers belonging to λ' be

$$m_1', m_2', \cdots\cdots\cdots m'_{\nu'-1}, m'_{\nu'} = m';$$

let the numbers belonging to λ'' be

$$m_1'', m_2'', \cdots\cdots\cdots m''_{\nu''-1}, m''_{\nu''} = m'';$$

etc., etc. Form a diagram corresponding to λ' by arranging m' dots in rows and columns so that there shall be ν' rows respectively of

$$m_1',\ m_2'-m_1',\ \cdots\cdots\ m'_{\nu'}-m'_{\nu'-1},$$

dots. The number of columns will then be m'_1. In one column there will be ν' dots. Let the number of dots in these m_1' columns be respectively

$$p_1', p_2', \cdots\cdots\ p'_{m'_1}.\text{*}$$

Next form a diagram corresponding to λ'' by arranging m'' dots in rows and columns so that there shall be ν'' rows respectively of

$$m_1'',\ m_2''-m_1'',\ \cdots\cdots\ m''_{\nu''}-m''_{\nu''-1},$$

dots; and let the number of dots in these m_1'' columns be respectively

$$p_1'',\ p_2'',\ \cdots\cdots\ p''_{m''_1}.$$

Proceed in the same way to form diagrams corresponding to the other roots of the characteristic equation of A.

Let now δ be the greatest common divisor of the p's. *Then if k contain δ, A can be generated by the repetition of an infinitesimal transformation of the special linear homogeneous group and not otherwise.* In particular, if m', m'', etc., have no common factor, A can be generated by the repetition of an infinitesimal transformation of the special linear homogeneous group.

If the transformation A can be generated by the repetition of an infinitesimal transformation of the special linear homogeneous group, A is the m^{th} power of a transformation of this group for any exponent m.

If δ_1 is the greatest common divisor of k and δ, then A is the m^{th} power of a transformation of the special linear homogeneous group for any exponent m that contains no factor of δ except a factor of δ_1; but for no other exponent is A

---

*If the numbers belonging to λ' are 5, 9, 13, 16, the diagram corresponding to λ' is

$$\begin{array}{lllll} * & * & * & * & * \\ * & * & * & * & \\ * & * & * & * & \\ * & * & * & & \end{array}$$

and $\qquad p_1'=p_2'=p_3'=4,\quad p_4'=3,\quad p_5'=1,$

the m^{th} power of a transformation of the special linear homogeneous group. In particular if m is relatively prime to n, the number of variables, A is the m^{th} power of a transformation of the special linear homogeneous group.

If A cannot be generated by the repetition of an infinitesimal transformation of the special linear homogeneous group, the m^{th} power of A can be generated thus, provided m contains $^{\delta}/_{\delta_1}$ (the quotient of δ by the greatest common divisor of δ and k), but not otherwise.

March 4, 1896.

NOTES.

A REGULAR meeting of the American Mathematical Society was held in New York on Saturday afternoon, February 29, at three o'clock, the President Dr. HILL, in the chair. There were eleven members present. On the recommendation of the Council, Mr. HARRY BROWN EVANS, University of Pennsylvania, Philadelphia, Pa., was elected to membership. The following papers were read:

(1) Professor M. I. PUPIN: "On the integral of the differential equation

$$a\frac{d^2y}{dt^2} + b\frac{dy}{dt} = c\frac{d^2y}{dx^2}$$

under certain boundary conditions occurring in the theory of long electrical waves."

(2) Professor E. H. MOORE: "A two-fold generalization of Fermat's theorem."

In the absence of Professor Moore his paper was read by the Secretary.

AMONG the courses for the summer semester at Berlin are the following—Professor Fuchs: Introduction to the theory of differential equations; Introduction to the theory of functions;—Professor Frobenius: Theory of numbers (second course); Theory of determinants;—Professor Schwarz: Analytical geometry; Applications of the theory of the elliptic functions; Hyper-geometric series; Mathematical colloquium;—Professor Knoblauch, Theory of the elliptic functions; Theory of curved surfaces; Theory of space curves; —Professor Hettner: Introduction to the theory of infinite series, products, and continued fractions;—Professor Hensel: Concerning the transcendentalism of e and π; Higher algebra (group and substitution theories);—Dr. Schlesinger: Geometry of three dimensions (synthetic and analytic); Differential calculus and introduction to analysis; Exercises in the differential calculus;—Dr. Kötter: Integral calculus; Focal properties of surfaces of the second order; Exercises in the integral calculus;—Dr. Hoppe: Integral calculus; Analytical mechanics;—Dr. Arons: Application of the differential and integral calculus to physical and chemical problems;—Dr. Glan : Quaternions.

AMONG the courses for the summer semester at Göttingen are the following:—Professor Schering: Mechanics;—Professor Klein: Special topics in the theory of numbers; Concerning problems in applied mechanics;—Professor Schur: Numerical calculation and applications to the method of least squares;—Professor Burkhardt; Elliptic functions;—Professor Hilbert; Theory of ordinary differential equations; Concerning the squaring of the circle;—Professor Schönflies: Analytical geometry; Special topics in the theory of functions;—Dr. Sommerfeld: Theory of surfaces, Exercises in the theory of surfaces;—Dr. Bohlmann: Differential calculus.

THE French Association for the Advancement of Science holds its annual meeting this year at Tunis, April 1–4; E. Collignon is Chairman of Sections I and II (mathematics, mechanics, and astronomy).

A WORK of more than ordinary interest and originality has just been issued from the press of B. G. Teubner. It is the first volume of the *Geometrie der Berührungstransformationen, dargestellt von* SOPHUS LIE *und* G. SCHEFFERS. The authors give the following synopsis of the subjects treated in this volume:

"During the years 1869–71, Lie developed a geometrical theory of contact transformations and explained by means of it the mutual connection and interrelation between projective and metrical geometry. At the same time he showed the fertility of this theory by numerous applications to partial differential equations of the first and second orders. The present work is intended to give a full presentation of these theories.

It has been attempted to present the subject in such a form as to make it intelligible not only to the professional mathematician but even to students who are acquainted merely with the more elementary branches of higher mathematics.

The first volume is divided into three parts. In the *first part* it is shown how, in the course of time, the idea of point transformations was introduced and applied in geometry. Particular attention is here given to the first contact transformation ever introduced into mathematics, viz., dilatation, which really lies at the basis of Huygens's theory of waves. Next the transformations by means of pedals and reciprocal polars are treated. After these and other examples, the general idea of the *contact transformation in the plane* is defined both analytically and geometrically and its theory is fully developed.

The importance of the theory of *infinitesimal* contact transformations is finally illustrated by an elegant and difficult problem: to determine the form of the element of arc for all those surfaces on which the system (Schaar) of geodesic circles admits a continuous system of contact transformations or a discrete number of point transformations.

While the first part contains a theory of the transformations of line elements in the plane, the *second part* is intended to give a *geometry of line elements in space.* This geometry comprises the *Pfaffian equations* and expressions closely connected with the *null-systems*, and the *Mongian equations* $\Omega(x, y, z; dx, dy, dz) = 0$ to which Plücker's *line complexes* are intimately related. All these theories appear here from a new point of view, as parts of the same whole. To illustrate the wide applicability of these views a theory of the *tetrahedral complexes* is given and a number of important categories of *partial differential equations* of the second order are integrated.

The central point of the discussions of the second part will again be found in its last chapter where an important polar relation between two spaces is investigated, each space containing a certain line complex whose lines correspond to the points of the other space. This establishes a correspondence between the surface elements of the two spaces which is really a *contact transformation in space transforming straight lines into spheres and asymptotic curves into lines of curvature.* This at once transforms Plücker's line geometry into a sphere geometry.

The *third part* introduces the *geometry of surface elements.* First, Lagrange's theory of *partial differential equations of the first order* is developed geometrically, according to Monge. It is then shown how much this theory gains in simplicity through the introduction of the surface element as a fundamental idea. Then follows the consideration of partial differential equations of the first order admitting infinitesimal point transformations; finally, a series of categories of such equations, of special interest in geometry, is investigated."

NEW PUBLICATIONS.

HIGHER MATHEMATICS.

BOLYAI (J.). The science absolute of space, independent of the truth or falsity of Euclid's axiom XI (which can never be decided *a priori*). Translated from the Latin by G. B. Halsted. 4th edition. Austin, Halsted, 1896. 12mo. 30 and 72 pp. Cloth. $1.00

BONETTI (E.). Alcune questioni sui contatti sferici, precedute da una esposizione sui centri, assi e piani di similitudine. Messina, 1895. 8vo. 12 pp.

CAYLEY (A.). Collected mathematical papers. (In 13 volumes.) Vol. 9. Cambridge University Press (Macmillan). 4to. 16 and 621 pp. $6.50.

DAUGE (F.). Cours de méthodologie mathématique. 2e édition, revue et augmentée. Gand, 1896. 8vo. 10 and 525 pp., 12 plates. Fr. 13.00

DURÈGE (H.). Elements of the theory of functions of a complex variable with especial reference to the methods of Riemann. Authorized translation from the fourth German edition by G. E. Fisher and I. J. Schwatt. Philadelphia, Fisher & Schwatt, 1896. 8vo. $2.50.

GAMBARDELLA (F.). Elementi di calcolo infinitesimale. 2a edizione. Livorno, Giusti, 1895. 8vo. 8 and 517 pp. Fr. 9.00

GERNET (M.). Ueber Reduktion hyperelliptischer Integrale. [Diss.] Heidelberg, 1895. 8vo. 40 pp.

GRANE (N.). Ueber Kurven mit gleichartigen successiven Developpoiden. Lund, 1894. 8vo. 71 pp. Mk. 2.50

GRIESS (J.). See KLEIN (F.).

HACK (F.). Beiträge zur Anwendung der Gruppentheorie auf kubische und biquadratische Gleichungen. [Diss.] Tübingen, 1895. 8vo. 36 pp.

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HALSTED (G. B.). See BOLYAI (J.).

HANSEN (P.C.V.). Et Brudstykke af den analytiske Plangeometri. Kjöbenhavn, 1895. 4to. 48 pp. (Lithographed.) Mk. 1.80

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KÖNIGSBERGER (L.). Hermann von Helmholtz's Untersuchungen über die Grundlagen der Mathematik und Mechanik. Heidelberg, 1895. 4to. 70 pp.

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Morawetz (J.). Zur analytischen Geometrie des Punktes und der Geraden. [Progr.] Salzburg, 1895. 8vo. 24pp.

Nüsslein (T.). Ueber die ebenen Configurationen C_3. Leipzig, 1895. 8vo. 36 pp., 2 plates. Mk. 1.80

Rath (E.). Die Grundformeln der allgemeinen Kurven- und Flächentheorie im nichteuklidischen Raum. [Diss.] Tübingen, 1894. 8vo. 42 pp.

Schlegel (V.). Die Grassmann'sche Ausdehnungslehre. Ein Beitrag zur Geschichte der Mathematik in den letzten fünfzig Jahren. (*Zeitschrift für Mathematik und Physik*, Vol. 41 (1896), historisch-literarische Abteilung, pp. 1-21.) 8vo.

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Schotten (H.). Der Koördinatenbegriff und die analytische Geometrie der Kegelschnitte. Berlin, Grote, 1895.

II. ELEMENTARY MATHEMATICS.

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Barberis ed **Ottini**. Sviluppo, costruzione e misurazione dei solidi geometrici. Torino, 1896. 8vo. 14 plates. Fr. 1.00

Bremiker (C.). Logarithmisch-trigonometrische Tafeln. Neu bearbeitet von T. Albrecht. 12te Ausgabe. Berlin, 1895. 8vo. 18 and 598 pp. Mk. 4.20

Büttner (A.). Elemente der Buchstabenrechnung und Algebra; nebst Anhang: Logarithmentafeln für die Zahlen von 1 bis 10,000. 12te Auflage. Bielefeld, 1896. 8vo. 4 and 192 pp. Cloth. Mk. 3.20

Capel (A. D.). Common sense Euclid. Part I: Books 1 and 2, with 300 problems. 4th edition, revised and enlarged. Part II: Books 3 and 4, with 300 problems. London, Abbott, 1896. 12mo. 156 and 140 pp. 1s. 6d. and 2s.

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HALL (L.). The elements of algebra. New York, American Book Co., 1895. 12mo. 368 pp. Half leather. $1.00

HANSON (C.). Elementaer Plangeometri. 2te udgave. Hjörring, 1895. 8vo. 112 pp. Cloth. Mk. 2.20

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Jahrbuch für das deutsche Versicherungswesen 1896. Lebens-, Renten- und Unfallversicherung nebst vergleichenden Tabellen der Prämien deutscher Lebensversicherungs-Gesellschaften. Herausgegeben von J. Neumann. Berlin, 1896. 12mo. 8 and 536 and 48 pp. Cloth. Mk. 10.00

Loudon (W. J.). An elementary treatise on rigid dynamics. New York, Macmillan, 1896. 8vo. 10 and 236 pp. Cloth. Mk. 2.25

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Maneuvrier (G.). Traité élémentaire de mécanique rationnelle et appliquée. Paris, Hachette, 1896. 16mo. 320 pp. Fr. 4.00

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Articles for insertion should be addressed to the

BULLETIN OF THE AMERICAN MATHEMATICAL SOCIETY,

41 East 49th Street, New York City.

The following dates have been fixed for the meetings of the Society:

			P. M.				P. M.
Sat.,	Oct. 26,	1895,	3:00	Sat.,	Feb. 29,	1896,	3:00
"	Nov. 30,	"	"	"	Mar. 28,	"	"
Fri.,	Dec. 27,	"	"	"	April 25,	"	"
Sat.,	Jan. 25,	1896,	"	"	May 23,	"	"

[illegible]

8

BULLETIN

OF THE

AMERICAN MATHEMATICAL SOCIETY.

Continuation of the Bulletin of the New York Mathematical Society.

A Historical and Critical Review
of Mathematical Science.

EDITED BY
THOMAS S. FISKE, ALEXANDER ZIWET, FRANK MORLEY.

2nd Series, Vol. II. No. 8.

May, 1896.

NEW YORK:
Published for the Society
By MACMILLAN & CO.
1896.

Entered at the Post-office at New York, N. Y., as second-class matter.
Published Monthly, except August and September.
Whole No. 48. $5.00 a year.

THE ARITHMETIZING OF MATHEMATICS.*

An Address delivered at the public meeting of the Royal Academy of Sciences of Göttingen, November 2nd, 1895.

BY PROFESSOR FELIX KLEIN.

THOUGH the details of mathematical science, by their very nature, elude the comprehension of the layman, and therefore fail to arouse his interest, yet the mathematician may profitably indicate certain general points of view from which he surveys his science, especially if these points of view determine his attitude to kindred subjects. I propose therefore on the present occasion to explain my position in regard to an important mathematical tendency which has as its chief exponent Weierstrass, whose eightieth birthday we have lately celebrated. I refer to the *arithmetizing* of mathematics. Some account of this tendency and its origin may be given by way of preface.

The popular conception of mathematics is that of a strictly logically coördinated system, complete in itself, such as we meet with, for instance, in Euclid's geometry; but as a matter of fact, modern mathematics in its origin was of a totally different character. With the contemplation of nature as starting point, and its interpretation as object, a philosophical principle, the principle of continuity, was made fundamental; and the use of this principle characterizes the work of the great pioneers, Newton and Leibnitz, and the mathematicians of the whole of the eighteenth century—a century of discoveries in the evolution of mathematics. Gradually, however, a more critical spirit asserted itself and demanded a logical justification for the innovations made with such assurance, the establishment, as it were, of law and order after the long and victorious campaign. This was the time of Gauss and Abel, of Cauchy and Dirichlet. But this was not the end of the matter. Gauss, taking for granted the continuity of space, unhesitatingly used space intuition as a basis for his proofs; but closer investigation showed not only that many special points still needed proof, but also that space intuition had led to the too hasty assumption of the generality of certain theorems which are by no means general. Hence arose the demand for exclusively arithmetical methods of proof; nothing shall be accepted as a part of the science unless its rigorous truth can be clearly demonstrated by the ordinary operations of analysis. A glance at the more modern text-books of the differential and integral calculus suffices to show the great change in method; where formerly a diagram served as proof, we now find continual discussions of quantities which become smaller than, or which can be taken smaller than, any given small quantity. The continuity of a variable, and what it implies, or does not imply, are discussed, and a question is brought forward whether we can, properly speaking, differentiate or integrate a function at all. This is the Weierstrassian method in mathematics, the "Weierstrass'sche Strenge," as it is called.

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* Translated by ISABEL MADDISON, Bryn Mawr College.

Of course even this assigns no absolute standard of exactness; we can introduce further refinements if still stricter limitations are placed on the association of the quantities. This is exemplified in Kronecker's refusal to employ irrational numbers, and consequent reduction of mathematics to relations between whole numbers only; and in another way in the efforts made to introduce symbols for the different logical processes, in order to get rid of the association of ideas, and the lack of accuracy which creeps in unnoticed, and therefore not allowed for, when ordinary language is used. In this connection special mention must be made of an Italian mathematician, Peano, of Turin, to whom we are indebted for various interesting notes on other points.

Summing up all these developments in the phrase, *the arithmetizing of mathematics*, I pass on to consider the influence of the tendency here described on parts of the science outside the range of analysis proper. Thus, as you see, while voluntarily acknowledging the exceptional importance of the tendency, I do not grant that the arithmetized science is the essence of mathematics; and my remarks have therefore the two-fold character of positive approbation, and negative disapproval. For since I consider that the essential point is not the mere putting of the argument into the arithmetical form, but the more rigid logic obtained by means of this form, it seems to me desirable—and this is the positive side of my thesis—to subject the remaining divisions of mathematics to a fresh investigation based on the arithmetical foundation of analysis. On the other hand I have to point out most emphatically—and this is the negative part of my task—that it is not possible to treat mathematics exhaustively by the method of logical deduction alone, but that, even at the present time, intuition has its special province. For the sake of completeness I ought also to deal with the algorithmic side of mathematics, discussing the importance of symbolic methods, but as this subject does not appeal to me personally, I shall not enter upon it. It must be understood that I have not much that is new to say on special points; my object is rather to collect and arrange material already familiar, justifying its existence where necessary.

In the short time at my disposal I must content myself with presenting the most important points; I begin therefore by tracing the relation of the positive part of my thesis to the domain of geometry. The arithmetizing of mathematics began originally, as I pointed out, by ousting space intuition; the first problem that confronts us as we turn to geometry is therefore that of reconciling the results obtained by arithmetical methods with our conception of space. By this I mean that we accept the ordinary principles of analytical geometry, and try to find from these the geometrical interpretation of the more modern analytical developments. This problem, while presenting no special difficulty, has yet many ramifications, as I have had the opportunity of showing during the past year in a seminar devoted to this subject. The net result is, on the one hand, a refinement of the process of space intuition; and on the other, an advantage due to the clearer view that is hereby obtained of the analytical results considered, with the consequent elimination of the paradoxical character that is otherwise apt to attach itself to them. What is the most general idea of a curve, of a surface? What is meant by saying of a curve, etc., that it is "analytic" or "non-analytic?" These and similar questions must be thoroughly sifted and clearly explained. The next point is that we must subject the fundamental principles of geometry to a fresh investigation. As far as the theory of the matter is concerned, this might very well be done, as it was originally, on purely geometrical lines; but in practice on account of the overwhelming complications that present themselves, recourse must be had to the processes of analysis, that is to the methods of analytical geometry. The investigation of the formulæ by means of which we represent the different forms in space (that is, the so-called non-Euclidian geometry, and all that is connected with it) disposes of only one side, and that the more obvious one, of the inquiry; there still remains the more important question: What justification have we for regarding the totality of points in space as a number-manifoldness in which we interpolate the irrational numbers in the usual manner between the rational numbers arranged in three directions? We ultimately perceive that space intuition is an inexact conception, and that in order that we may subject it to mathematical treatment, we idealize it by means of the so-called axioms, which actually serve as postulates. Kerry, who died at an early age, dealt with the philosophical side of these questions, and I agree with his results in the main and especially as regards his criticism of DuBois Reymond. Conversely this fresh determination of our conception of space has in its turn given rise to new refinements of our analytical ideas. We picture before us in space an infinite number of points and forms composed of them ; from this idea have sprung the fundamental investigations on masses of points and transfinite numbers with which G. Cantor has opened up new spheres of thought to arithmetical science. Finally it is much to be desired that full use should be made of the new point of view in the further exposition of geometry, especially infinitesimal geometry ; this result will be most easily attained by treating the subject analytically. Of course, I do not mean by this a blind calculation with x, y and z, but merely a subsidiary use of these quantities when the question concerns the precise determination of boundary conditions.

From this outline of the new geometrical programme you see that it differs greatly from any that was accepted during the first half of this century, when the prevailing tendencies led to the development of projective geometry, which has long been established as a permanent constituent of our subject. Projective geometry has opened up for us with the greatest ease many new tracts of country in our science, and has been rightly called a royal road to its own particular branch of learning; our new road is on the contrary arduous and thorny, and unremitting care is needed to clear a way through the obstacles. It leads us back to what is more nearly the geometry of the ancients, and in the light of our modern ideas we learn to understand precisely the true nature of the latter, as Zeuthen has lately shown in the most brilliant manner.

Moreover we must introduce the same process of reasoning into mechanics and mathematical physics. To avoid going too much into detail I will merely explain this by two examples. Throughout applied mathematics, as in the case of space intuition, we must idealize natural objects before we can use them for purposes of mathematical argument; but we find continually that in one and the same subject we may idealize objects in different ways, according to the purpose that we have in view. To mention only a single instance, we treat matter either as continuous throughout space, or as made up of separate molecules, which we may consider to be either at rest or in rapid motion. How and to what degree are these different hypotheses equivalent in regard to the mathematical consequences that can be deduced from them? The earlier expositions of Poisson and others, as also the developments of the Kinetic Theory of Gases, are not sufficiently thorough in this respect for the modern mathematician; the problem requires to be investigated afresh ab initio. I expect that a publication by Boltzmann, which will shortly appear, will contain some interesting conclusions on this subject.

Another question is this: Practical physics provides us plentifully with experimental results, which we unconsciously generalize and adopt as theorems about the idealized objects. The existence of the so-called Green's function on any closed surface with an arbitrarily chosen pole, corresponding to the fact in electricity that a conductor under the influence of a charged point is in a state of electrical equilibrium, belongs to this category; as also the theorem that every finite elastic body is capable of an infinite series of harmonic oscillations, and my deduction of the fundamental propositions of Riemann's Theory of Abelian Functions from our knowledge of the electric currents started on any conductor when the poles of a galvanic battery are applied to it. Are these indeed, taken in the abstract, exact mathematical theorems, or how must they be limited and defined in order that they may become so? Mathematicians have successfully sought to investigate this; first, C. Neumann and Schwarz with their theory of Potential, and later the French school, following on the lines of the German, with the result that the theorems taken from physics have been shown to hold good to a very considerable extent. You see here what is the precise object of these renewed investigations; not any new physical insight, but abstract mathematical argument in itself, on account of the clearness and precision which will thereby be added to our view of the experimental facts. If I may use an expression of Jacobi's in a somewhat modified sense, it is merely a question of intellectual integrity, "die Ehre des menschlichen Geistes."

After expressing myself thus it is not easy, without running counter to the foregoing conclusions, to secure to intuition her due share in our science; and yet it is exactly on this antithesis that the point of my present statements depends. I am now thinking not so much of the cultivated intuition just discussed, which has been developed under the influence of logical deduction and might almost be called a form of memory; but rather of the naive intuition, largely a natural gift, which is unconsciously increased by minute study of one branch or other of the science. The word intuition (Anschauung) is perhaps not well chosen; I mean it to include that instinctive feeling for the proportion of the moving parts with which the engineer criticises the distribution of power in any piece of mechanism he has constructed; and even that indefinite conviction the practiced calculator possesses as to the convergence of any infinite process that lies before him. I maintain that mathematical intuition—so understood—is always far in advance of logical reasoning and covers a wider field.

I might now introduce a historical excursus, showing that in the development of most of the branches of our science, intuition was the starting point, while logical treatment followed. This holds in fact, not only of the origin of the infinitesimal calculus as a whole, but also of many subjects that have come into existence only in the present century. For example, I may remind you of Riemann's Theory of the Functions of a Complex Variable; and I am glad to add also that the Theory of Numbers, a subject which for a long time seemed to be most unsuited for intuitive methods of treatment, appears to have received a fresh impetus from the application of intuition in the hands of Minkowski and others. After this it would be a matter of great interest to trace from the present standpoint the development, not of particular mathematical subjects, but of the individual mathematician; but in regard to this it must suffice to mention that the two most active mathematical investigators of the present day, Lie in Leipzig, and Poincaré in Paris, both originally made use of intuitive methods. But all this, if I pursued it further, would lead us too much into detail, and finally bring us only to particular cases. I prefer to sketch the every day results of this somewhat refined intuition, as regards the quantitative, rather than the merely arithmetical or constructive, treatment of physical or technical problems. Let me refer again to the two examples from the theory of electricity already adduced; any physicist would be able to trace, without further difficulty, and with tolerable accuracy, the form of the surface of Green's function, or, in the second experiment, the shape of the lines of force in a given case. Again, consider any given differential equation, I will say, to take the most simple instance, a differential equation of the first order in two variables. Most probably the analytical method of solution fails; nevertheless we can at once find graphically the general form of the integral curves, as has recently been done for the renowned differential equation of the Problem of Three Bodies by Lord Kelvin, a master in the art of mathematical intuition. The question in all such cases, to use the language of analysis, is one of interpolation, in which less stress is laid on exactness in particular details than on a consideration of the general conditions. I will once more emphasize the fact that in stating all our laws of nature, or in trying to formulate mathematically any actual occurrence, the art lies in making a similar use of interpolation; for we have to consider the simple laws connecting the essential quantities, apart from the multitude of fortuitous disturbances. This is ultimately what I have termed above the process of idealization. Logical investigation is not in place until intuition has completed the task of idealization.

I beg that you will consider these remarks as a description, not as an explanation, of what actually occurs. The mathematician can do no more than state the character of each particular psychical operation from observations of his own mental process. Perhaps some day physiology and experimental psychology will enable us to draw more accurate conclusions as to the relation between the processes of intuition and those of logical thought. The great differences shown by observations of different individuals confirm the supposition that it is indeed a question of distinct, that is, not necessarily connected, mental activities. Modern psychologists distinguish between visual, motor and auditory endowments; mathematical intuition, as above defined, appears to belong more closely to the first two classes, and the logical method to the third class. In common with many of my fellow mathematicians I gladly welcome these investigations which psychologists have only just undertaken, for it is to be hoped that with the increase of accurate information about the psychological conditions of mathematical thought and their particular varieties, many of the differences of opinion which necessarily remain unsettled at present will disappear.

I must add a few words on mathematics from the point of view of pedagogy. We observe in Germany at the present day a very remarkable condition of affairs in this respect; two opposing currents run side by side without affecting one another appreciably. Among the teachers in our Gymnasia the need of mathematical instruction based on intuitive methods has now been so strongly and universally emphasized that one is compelled to enter a protest, and vigorously insist on the necessity for strict logical treatment. This is the central thought of a small pamphlet on elementary geometrical problems which I published last summer. Among the university professors of our subject exactly the reverse is the case; intuition is frequently not only undervalued, but as much as possible ignored. This is doubtless a consequence of the intrinsic importance of the arithmetizing tendency in modern mathematics. But the result reaches far beyond the mark. It is high time to assert openly once for all that this implies, not only a false pedagogy, but also a distorted view of the science. I gladly yield the utmost freedom to the preferences of individual academic teachers, and have always discouraged the laying-down of general rules for higher mathematical teaching, but this shall not prevent me from saying that two classes at least of mathematical lectures must be based on intuition; the elementary lectures which actually introduce the beginner to higher mathematics—for the scholar must naturally follow the same course of development on a smaller scale, that the science itself has taken on a larger—and the lectures which are intended for those whose work is largely done by intuitive methods, namely, natural scientists and engineers. Through this one-sided adherence to logical form we have lost among these classes of men much of the prestige properly belonging to mathematics, and it is a pressing and urgent duty to regain this prestige by judicious treatment.

To return to theoretical considerations, the general views which I uphold in regard to the present problems of mathematical science need scarcely be specially formulated. While I desire in every case the fullest logical working out of the material, yet I demand at the same time an intuitive grasp and investigation of the subject from all sides. Mathematical developments originating in intuition must not be considered actual constituents of the science till they have been brought into a strictly logical form. Conversely, the mere abstract statement of logical relations cannot satisfy us until the extent of their application to every branch of intuition is vividly set forth, and we recognize the manifold connections of the logical scheme, depending on the branch which we have chosen, to the other divisions of our knowledge. The science of mathematics may be compared to a tree thrusting its roots deeper and deeper into the earth and freely spreading out its shady branches to the air. Are we to consider the roots or the branches as its essential part? Botanists tell us that the question is badly framed, and that the life of the organism depends on the mutual action of its different parts.

DARBOUX'S MEMOIR ON CYCLIQUES.

Sur une Classe Remarquable de Courbes et de Surfaces Algébriques et sur la théorie des Imaginaires, par GASTON DARBOUX, Doyen de la Faculté des Sciences de Paris. Second tirage. Paris, A. HERMANN, 1896.

IT is a pity that M. Darboux did not make some additions to this work since its first publication in 1873. He is so full of ideas in a number of mathematical directions that it is a cause for regret that he has not devoted some more time to a subject which offered him once such a fruitful field for original investigation. However, those who know the work will be glad to make acquaintance with it again, and others who are tired of conic sections and quadrics may be gratified by finding in it a somewhat similar but still novel field of investigation.

Darboux calls the curves and surfaces which he treats of cycliques and cyclides, respectively. The latter word has been generally adopted in English as a name for the surfaces, but the former has been replaced by bicircular quartics and sphero-quartics, which two designations conveniently distinguish between the plane and spherical curves. The name cyclide had already been used for a particular form of the surface by Dupin, and soon came to be adopted generally in its present sense, while the different names of the curves arose from their being studied independently in Great Britain and France. At one time, from about 1865 until early in the seventies, these curves and surfaces were studied enthusiastically in France by Darboux, Laguerre, de la Gournerie, Moutard, Mannheim and others, while they received attention in England at the hands of Crofton and Clifford, and in Ireland secured very full and adequate treatment from Casey. All these mathematicians worked at the same time and nearly entirely independently of each other. Thus Darboux's book contains much that has been more fully worked out elsewhere, and, besides, of course, it has no references to many striking results that have been arrived at in this branch of geometry, as, for instance, Casey's ingenious method of rectification of the bicircular quartics and sphero-quartics. But these wants are made up for by Darboux's methods of investigation and by his curious paths of inquiry, from which new views and sidelights on the subject are continually obtained. It is this, as well as the incidental solutions of many problems not directly connected with the subject, that gives a charm to the work, and only those who have a preference for formal and logically ordered treatises will be displeased with its discursive character.

M. Darboux begins by considering the transformation of inversion, and in addition to the ordinary properties of this process shows that foci and focal curves, when inverted, possess the same relation to the inverted curve. Then he shows that every *cyclique* is transformed into a similar curve by inversion. Thus for an origin outside of its plane, the bicircular quartic, or plane quartic with the two circular points at infinity as nodes, is transformed into a sphero-quartic, or curve of intersection of a sphere and a quadric. In this way he is enabled to select the most suitable curves for his first investigation. Choosing the sphero-quartic he makes use of the known result that four cones of the second order can be described passing through the curve. Since the tangent planes of the cones intersect the sphere in circles having double contact with the curve, he thus deduces the four-fold generation of the sphero-quartic as the envelope of small circles orthogonal to a given circle J and having their centres on a sphero-conic F. Then by geometrical considerations the four circles such as J are shown to be mutually orthogonal and the four sphero-conics such as H to be confocal, and further the focal curves are shown to lie on four mutually orthogonal spheres having their centres at the vertices of the cones. In this way M. Darboux establishes these and many other properties of the curves by means of previously known results. M. Darboux obtains a number of properties of the bicircular quartics, sphero-quartics and allied curves of higher orders by the use of imaginaries. In particular we may mention the differential equation of the system of confocal bicircular quartics, viz:

$$\frac{du}{\sqrt{f(u)}} \pm \frac{dv}{\sqrt{f_1(v)}} = 0.$$

where

$$f(u) = A(u-a_1)(u-a_2)(u-a_3)(u-a_4)$$
$$f_1(v) = B(v-b_1)(v-b_2)(v-b_3)(v-b_4),$$
$$u = x+iy, \quad v = x-iy,$$

and the anharmonic ratios of

$$f(u)=0, \quad f_1(v)=0$$

are equal.

This equation gives the complete theory of the arrangement of the 16 foci of the curve, for by the definition of a focus these points are found from $du=0$, $dv=0$. Thus we get $f(u)=0, f_1(v)=0$, which determine 16 points; and these from the equality of the anharmonic ratios must lie by fours on four circles.

By the use of imaginaries M. Darboux also obtains properties of curves comprised in the equation

$$R_1 R_2 R_3 \cdots = k\, r_1 r_2 r_3 \cdots ,$$

where $R_1, R_2, \cdots, r_1, r_2 \cdots$, are the distances from fixed points, and he shows that such curves can be written in this form in an infinite number of ways. Similar results are obtained for the sphere, and theorems concerning closed polygons formed by the imaginary generators of the sphere are proved. He thus gives a new demonstration of Poncelet's theorems for polygons of an even number of sides and arrives at an important theorem concerning polyhedra formed by generators of a quadric, viz: that if they form a polygon having its vertices on the curve of intersection with another quadric, then there exist an infinite number of polygons formed in the same way. M. Darboux observes that this result was given by Moutard in the *Nouvelles Annales de Mathématiques* in 1864. I think, however, that Steiner and Hart were acquainted with it previously. In 1846 Steiner gave in Crelle's Journal a theorem concerning polygons inscribed in a binodal quartic. This is to the effect that, if a certain condition is satisfied, an infinite number of polygons can be described, such that one set of alternate sides all pass through one node, while the other set of alternate sides pass through the other node. Now let the nodes be the circular points, then if we invert from a point outside the plane, the lines through the nodes become the imaginary generators of a sphere, and the curve becomes a sphero-quartic. Thus we see the identity of Steiner's theorem with that just mentioned. I may notice that Professor A. R. Forsyth, who has treated this problem from an analytical point of view, makes no mention of those who had gone before him. (*Proceedings of the London Mathematical Society*, vol. 14, p. 35.)

M. Darboux further notices that the same property exists when we substitute for the circumscribing quadric quadrics inscribed in the developable formed by the inscribed quadric and one of the circumscribing quadrics, or which is the same thing, if we suppose the vertices to lie on confocal quadrics; and he observes that this proposition seems new. Hart, however, was previously aware of the fact that there could be an infinite number of geodesic polygons circumscribed about a line of curvature of a quadric so that these vertices lie on other lines of curvature. Now, if we suppose the quadric passing through the inscribed line of curvature to coincide with the given one, the geodesics become simply the generators. Thus Darboux's result is comprised in Hart's theorem. (*Cambridge and Dublin Mathematical Journal*, vol. 4, p. 192.)

Proceeding to consider the surface M. Darboux makes use of analysis. With the help of the inscribed quadrics, viz.: quadrics such as V when the equation of the surface takes the form $S^2 = V$, where S is a sphere, he deduces the generation of the cyclide as the envelope of a sphere orthogonal to a given sphere J and having its centre on a quadric F. Further he shows that this generation is fivefold, the five spheres such as J being mutually orthogonal, and the five quadrics such as F being confocal, while the five spheroquartics FJ are the focal curves. This mode of treatment, it may be observed, has been adopted by Dr. Salmon in his geometry of three dimensions. The remarkable form assumed by the equation of confocal cyclides in terms of the five spheres is worth noticing. If X, Y, Z, U, V represent the squares of the tangents drawn from a point to the spheres divided by the radius of each sphere, then $X^2 + Y^2 + Z^2 + U^2 + V^2 = 0$ identically; and the equations

$$\frac{X^2}{\lambda-a_1}+\frac{Y^2}{\lambda-a_2}+\frac{Z^2}{\lambda-a_3}+\frac{U^2}{\lambda-a_4}+\frac{V^2}{\lambda-a_5}=0$$

represent confocal cyclides. The remarkable fact that spheres such as $l_i X + m_i Y + n_i Z + p_i U + q_i V = 0$, cut orthogonally if $\Sigma l_1 l_2 = 0$, shows at once that the three confocal cyclides drawn through a point cut orthogonally. There is in these equations an analogy to linear expressions passing through a point and cones, if we may call them so, in four dimensions. In fact just as bicircular quartics are the inverses of sphero-conics, so cyclides may be regarded as the inverses of the sphero-quadrics of four dimensions. This system of coördinates enables us to express the coördinates of a point on a cyclide in terms of two parameters. For solving for X in terms of the parameters of three surfaces of the system we get

$$X^2 = \frac{(a_1 - \lambda_1)\,(a - \lambda_2)\,(a_1 - \lambda_3)}{f'(a_1)},$$

where

$$f(\lambda) = (\lambda - a_1)(\lambda - a_2)(\lambda - a_3)(\lambda - a_4)(\lambda - a_5),$$

and similar values for Y, Z, U, V. Hence if we suppose λ_3 to be a constant for a given surface we obtain, from

$$X = \frac{1}{r}(x^2 + y^2 + z^2 - 2\,\alpha\,x + 2\,\beta\,y - 2\,\gamma^2 + \delta)$$

and similar equations, x, y, z expressed linearly in terms of the five radicals $\surd\{(a_i - \lambda_1)\,(a_i - \lambda_2)\}$. Thus by the theory of hyperelliptic integrals if we write

$$\frac{d\lambda_1}{\surd f(\lambda_1)} + \frac{d\lambda_2}{\surd f(\lambda_2)} = d\,u_1, \quad \frac{\lambda_1\,d\lambda_1}{\surd f(\lambda_1)} + \frac{\lambda_2\,d\lambda_2}{\surd f(\lambda_2)} = d\,u_2,$$

we have the coördinates expressed linearly in terms of the inverse hyperelliptic functions of u, v, called $al(u_1, u_2)_1$, $al(u_1, u_2)_2$, etc., by Weierstrass, just as the coördinates of a point on a quadriquadric curve can be expressed linearly in terms of sn$u$, cn$u$, dn$u$.

In the notes and additions, which comprise a considerable portion of the work, M. Darboux gives a number of miscellaneous theorems concerning curves and surfaces. In particular he studies the system of coördinates obtained by taking as variables the parameters of the tangents which can be drawn from a point to a given conic. Thus if the conic is $\beta^2 - 4\,\alpha\,\gamma = 0$, a tangent may be written

$$\alpha\,\rho^2 + \beta\,\rho + \gamma = 0,$$

so that if ρ_1, ρ_2 are the parameters of the tangents drawn from α, β, γ, we have

$$\alpha = -\frac{\beta}{\rho_1 + \rho_2} = \frac{\gamma}{\rho_1\,\rho_2}.$$

By means of these coördinates he obtains several interesting results regarding polygons. For instance, if a curve of the n^{th} degree pass through the intersection of the n tangents, $A_1, A_2 \cdots A_n$ with the n tangents $B_1, B_2, \cdots B_n$, its equation is

$$A_1\,A_2 \cdots A_n = k\,B_1\,B_2 \cdots B_n.$$

Since

$$A_i = \alpha a_i^2 + \beta a_i + \gamma = \alpha\,(a_i - \rho_1)(a_i - \rho_2) \text{ and } B_i = \alpha\,(b_i - \rho_1)(b_i - \rho_2),$$

this equation takes the form

$$\frac{\varphi\,(\rho_1)\;\varphi\,(\rho_2)}{\psi\,(\rho_1)\;\psi\,(\rho_2)} = k.$$

But this form can be assumed in a singly infinite number of other ways, as we see by writing it

$$\frac{\{\psi(\rho_1)+\lambda\varphi(\rho_1)\}}{\{k\lambda\psi(\rho_1)+\varphi(\rho_1)\}}\,\frac{\{\psi(\rho_2)+\lambda\psi(\rho_2)\}}{\{k\lambda\psi(\rho_2)+\varphi(\rho_2)\}}=\frac{1}{k}$$

Hence if a curve pass through the n^2 points where n tangents to a conic meet other n tangents, it must pass through an infinity of such systems of n^2 points. Further, if a curve of the n^{th} degree pass through all the points of intersection of $n+1$ tangents to the conic, it may be written

$$\frac{a_1}{A_1}+\frac{a_2}{A_2}+\cdots\frac{a_{n+1}}{A_{n+1}}=0$$

which becomes

$$\Sigma\frac{a_i}{(a_i-\rho_1)\ (a_i-\rho_2)}=0$$

or, if we multiply by $\rho_1-\rho_2$

$$\Sigma\frac{a_i}{a_i-\rho_1}=\Sigma\frac{a_i}{a_i-\rho_2}$$

that is

$$\frac{f(\rho_1)}{\varphi(\rho_1)}=\frac{f(\rho_2)}{\varphi(\rho_2)}$$

which may also be written

$$\frac{f(\rho_1)}{\varphi(\rho_1)+kf(\rho_1)}=\frac{f(\rho_2)}{\varphi(\rho_2)+kf(\rho_2)},$$

which is of the same form as the preceding with indeterminate k. Hence the curve will pass through all the vertices of an infinite number of polygons of $n+1$ sides circumscribed to the conic.

In this way M. Darboux proves the theorem originally obtained by Lüroth (*Math. Annalen* vol. 1.) viz: that if a quartic curve pass through all the points of intersection of five lines it cannot be the general curve of that degree, but must satisfy an invariant relation. M. Darboux might have noticed also that a general quartic curve cannot pass through the 16 points where four tangents of a conic meet four other tangents.* The invariant condition satisfied in this case is a problem suggested here for solution. Again by the use of these coördinates M. Darboux gives a very simple proof of Poncelet's theorem concerning polygons.

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* It may be observed that this result can also be put in the form : the twelve vertices and eight points of contact of two quadrilaterals circumscribed about a conic lie on a quartic curve which is not a general one.

It may be noticed that W. K. Clifford arrived at these results in precisely the same way (*Proceedings of the London Mathematical Society*, vol. 7, p. 29), and when he found that he had been anticipated by Darboux, expressed his opinion that this, viz: the work of Darboux that we are now considering, is a book which it is almost inexcusable in a geometer not to have read, marked, learned and inwardly digested.

There are many other interesting investigations in the book, especially a study at considerable length of the intersection of a sphere with a cyclide; an extension also of Ivory's theorem to confocal cyclides is worthy of notice. If A, B, C are three points on the cyclide and A', B', C' are three corresponding points on a confocal cyclide, M. Darboux shows that the relation

$$AB'.BC'.CB'=BA'.CB'.BC'$$

connects the distances between the points.

R. A. ROBERTS.

BESSEL FUNCTIONS.

A Treatise on Bessel Functions and their Applications to Physics. By ANDREW GRAY and G. B. MATHEWS. Macmillan & Co. 1895. 8vo, x+292 pp.

The transcendental functions to which Bessel's name has been attached are not only of the highest importance in mathematical physics, second perhaps only to the trigonometric and exponential functions, they are also of great interest to the student of pure mathematics both from the formal side and from the point of view of the theory of functions. There has, however, up to this time been no connected treatment of these functions in the English language, with the exception of the utterly inadequate treatment contained in the last sixty-five pages of Todhunter's book, *The Functions of Laplace, Lamé and Bessel*, published twenty years ago. The German monographs by C. Neumann and Lommel make no attempt to cover more than small portions of the subject, and the same is true to an even greater extent of the sections devoted to Bessel's functions in Heine's *Kugelfunctionen*, Basset's *Hydrodynamics*, Rayleigh's *Sound* and elsewhere. Messrs. Gray and Mathews have therefore filled a real gap in mathematical literature.

The authors make it clear in their preface that their own interest is chiefly for the applications to mathematical physics. They have, however, none of the intolerance for and dislike of pure mathematics which is unfortunately too often manifested. The authors' position may, perhaps, be best explained in their own words: "Some readers may be inclined to think that the earlier chapters contain a needless amount of tedious analysis. . . . . . As a matter of fact it will be found that little, if any, of the analytical theory included in the present work has failed to be of some use or other in the later chapters; and we are so far from thinking that anything superfluous has been inserted that we could almost wish that space would have allowed of a more extended treatment, especially in the chapters on the complex theory and on definite integrals."

The book consists of fifteen chapters, followed by a few pages of miscellaneous matter and tables. Of the fifteen chapters, eight (87 pages) are devoted to the theory, while the remaining seven (135 pages) treat of the applications. This separation of theory and application is by no means complete, much analysis which might easily have been included in the earlier chapters being left until needed, while a few physical problems are introduced for the sake of illustration into the chapters on the theory of Bessel's functions. We think there can be no doubt that the authors have been wise in carrying this separation at least as far as they have done, as it conduces greatly to the clearness of the book, and makes it, what is very important, a convenient book of reference. The book might have been made even more useful in this respect than it is now by a somewhat extended collection of formulæ, including in particular a table of definite integrals involving Bessel's functions, which could have been placed with the numerical tables at the end of the book.

CHAPTER I: *Introductory* shows how Bessel's functions naturally present themselves in two simple physical and in one astronomical problem. The problems chosen (vibration of a heavy chain hanging from one end, cooling of a cylinder, Kepler's equation) are not only interesting in themselves; they have also a decided historic interest as being among the first problems in which Bessel's functions were used.*

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* There is a slight historical slip on p. 2, in the statement that after the problem of the heavy chain "the next appearance of a Bessel function is in Fourier's *Théorie analytique de la chaleur.*" The problem of the vibration of a circular membrane had been treated by Euler in 1764 (Cf. *Bulletin*, February, 1893, p. 108). A reference to Bessel's paper of 1818 might also well have been made.

CHAPTER II: *Solution of the Differential Equation.* The title of this chapter indicates the starting point chosen by the authors for the systematic study of Bessel's functions. The differential equation is solved by means of power series and a few simple relations between the Bessel's functions thus obtained are established. On p. 8 it is pointed out that the series for $J_{-n}(x)$ ceases to exist ("becomes unintelligible" is the unfortunate expression used) when n is a positive integer, and on p. 12 the function is defined in this case by the formula: $J_{-n}(x)=(-1)^n J_n(x)$, even the mere statement of the reason for this definition (viz.: to preserve the continuity with regard to n) being relegated to a footnote. The treatment of such a fundamental point as this, which would not have covered half a page, ought surely to have been given. We are glad to find the Bessel's function of the second kind[*] $Y_n(x)$ introduced here by means of power series; the interesting, but for the beginner artificial, form in terms of Bessel's functions of the first kind is given later (pp. 21–23).

CHAPTER III: *Functions of Integral Order. Expansions in series of Bessel Functions.*—This chapter opens with Schlömilch's elegant definition of Bessel's functions of the first kind of integral orders as the coefficients in the expansion of $e^{\frac{x}{2}(t-1/t)}$ according to ascending and descending powers of t. Here as elsewhere in the book the authors have adopted a half-way course with regard to mathematical rigor which can perhaps be justified when we consider the class of readers for which the book is written. Thus they usually establish the convergency of the series or integrals with which they deal, but leave the question as to whether the series or integrals converge to the right value, and also questions as to rearranging the terms of a series, differentiating a series, etc., almost, if not quite, unconsidered. It should, however, be added that the authors sometimes regard this method of obtaining results as merely heuristic and establish their results when once found by other methods.

CHAPTER IV: *Semiconvergent Expansions* might well have been postponed until later, as at this point only Bessel's functions of the first kind can be treated at all, and that in a very imperfect way. The term *semiconvergent* is here used in its proper sense, a series of this sort being divergent, but such that for certain values of the argument and of n

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[*] The authors have seen fit to attach Neumann's name to this function, although in the case $n=0$ the function had been used by Euler.

the sum of its first n terms gives an approximation to the function we wish to represent. The use of the word in this sense is more than forty years old, and it is greatly to be regretted that of late years the term has been used in English in a totally different sense, viz.: non-absolutely or conditionally convergent. After the general solution u_r of Bessel's equation has been developed into a semiconvergent series the statement is made (p. 37): "It is to be observed ..... that it is not to be inferred that the expression above given for u_r reflects in any adequate way the *functional* properties of an exact solution ..... These considerations, however, do not debar us from employing the expressions u_r up to a certain point for approximate numerical calculation ....." The distinction here made between the numerical values of a function and its "functional properties" is one which would, we think, be found in the minds of a very large number of people. There exists, however, no such distinction in reality, as a function is nothing more nor less than what might be called an ideal table of numerical values, and any properties of a function are merely propositions concerning these numerical values. The peculiarity of a semiconvergent series is that it does not permit us, like the ordinary representations of functions, to compute these numerical values to any required degree of accuracy, but merely to a certain limited degree of accuracy. The subject of semiconvergent series together with the allied subject of the asymptotic values of functions is one with which most readers of this *Treatise* are sure to be so unfamiliar, and it is at the same time a subject of such importance both practically and theoretically, that we think a more extended and more accurate treatment might well have been given of it.

CHAPTER V: *The Zeros of the Bessel Functions* gives far too brief a treatment (a little over six pages) of one of the most important and fascinating questions connected with Bessel's functions. The chapter opens with Bessel's original proof that $J_0(x)$ has an infinite number of real roots, this proof being reproduced "on account not only of its historical interest, but of its directness and simplicity." The fact that the considerations here given not only establish the existence of an infinite number of real roots, but also serve to completely separate them, is not, however, touched upon. Stokes's method of computing the large roots of $J_0(x)$ by means of semiconvergent series is also treated in this chapter, but no discussion is given of the degree of accuracy which it enables us to attain. Another matter which might well have found a place, is the application of the method developed by Sturm,* or of some suitable modification of it † to Bessel's equation. At least a reference should have been made to Hurwitz's paper in Math. Ann., vol. 33.

CHAPTER VI: *Fourier-Bessel Expansions.* Certain definite integral formulæ are first deduced by means of Green's theorem. Then follows a simple problem in the flow of heat which leads up to the development of an arbitrary function in the form: $\sum_{s=1}^{s=\infty} A_s J_0(\lambda_s r)$ where the quantities λ_s are the roots of the equation $\lambda k J_0'(\lambda) + h J_0(\lambda) = 0$. Other developments of a similar nature follow. The work in this chapter is merely formal, no questions of convergency being discussed.

CHAPTER VII: *Complex Theory.* We agree with the authors in regretting that they did not feel able to devote more space to this interesting subject. Bessel's functions are treated in this chapter by means of their definition as definite integrals taken along paths in the complex plane. The functions with pure imaginary argument are here introduced, and the chapter closes with Lipschitz's strict treatment of the semiconvergent expansions already given.

CHAPTER VIII: *Definite Integrals Involving Bessel Functions.* The first half of this chapter is devoted to a clear and well-arranged exposition of the most fundamental parts of this side of the subject which seems to have appealed strongly to the authors. The last half of the chapter deals with the expression of an arbitrary function by means of a remarkable double integral involving Bessel's functions. This integral, which was in substance discovered by C. Neumann, belongs to an extended class of integrals of which Fourier's integral may serve as a type. In fact this integral bears precisely the same relation to the expansion in terms of Bessel's functions considered in Chapter VI. that Fourier's integral bears to Fourier's series.‡ This fact is unfortunately not brought out by the authors although the similarity to Fourier's integral is mentioned.

CHAPTER IX: *The Relation of the Bessel Functions to*

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* Liouville's Journal, vol. I.

† See, for instance, Riemann-Hattendorf's *Partielle Differentialgleichungen*, p. 266-269, where, however, only the simplest case is considered and the presentation is somewhat faulty.

‡ This analogy seems the most natural, at least for a treatise on Bessel's functions. It is, however, also true, as has been pointed out by C. Neumann and Mehler, that one may regard this integral as having the same relation to the ordinary development in spherical harmonics, that Fourier's integral bears to Fourier's series.

Spherical Harmonics. In view of the physical tendencies of the book it might perhaps have been well, instead of giving a purely analytical treatment of this subject, to develop it in connection with the fact that cylinder coördinates may be regarded as a limiting case of polar coördinates in space. The treatment given, however, brings out clearly the analytic relations.

Before going on to the portion of the book in which the applications of Bessel's functions are taken up let us look back at the theoretical chapters we have just been considering in a somewhat more comprehensive way than we have yet done. While the treatment is too incomplete and even at times inaccurate to satisfy the pure mathematician, it is well adapted to the object the authors had in view of giving to the physicist a working knowledge of the more important properties of Bessel's functions. Even to the student of pure mathematics these chapters give, on account of the numerous sides of the subject on which they touch, an introduction to the theory of Bessel's functions which may fairly be considered satisfactory when supplemented by a certain amount of collateral reading. The presentation is clear and interesting and the references to the original memoirs useful.

The writers have avoided giving undue prominence to one side of the theory at the expense of other sides. There is, however, one matter of no little importance as it seems to us which the authors have hardly touched upon, namely, the differential equation which Bessel's functions satisfy. This equation to be sure is the starting point in Chapter II, but, when once its solutions have been expanded into series, these series, or the definite integrals which are proved equal to them, are made use of whenever any property of the functions is to be established.* Many of the properties of Bessel's functions may, however, be derived with ease and elegance directly from the differential equation. For this purpose one of the binomial forms (line 4, Chapter II, and (81) p. 35) is usually most convenient. The formulæ on pages 16 and 53 might well have been obtained in this way, while perhaps the most important matter which could best be so treated is the subject of the roots of Bessel's functions and allied questions. It should be borne in mind in introducing a student to the study of Bessel's functions that these are only an example of functions which occur frequently in mathematical physics, and which satisfy linear differential equations of the second order. With the exception of a few simple cases, which like Bessel's functions can be represented by definite integrals and simple series, no other means is usually available for discussing the properties of these functions, than the differential equation which they satisfy. It seems therefore advisable that in the case of Bessel's functions, although other methods are more or less applicable, the differential equation should not be altogether neglected.

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*Slight exceptions to this statement will be found in Chapters IV, and VI.

Another matter, a brief treatment of which might well have been included in this treatise, is the expression for the ratio of Bessel's functions of orders n and $n+1$ as a continued fraction.

Passing now to the chapters in which the applications to physical problems are considered, we are struck throughout by the very substantial knowledge of mathematical physics assumed on the part of the reader. This contrasts with the small amount of pure mathematics assumed in the earlier chapters. The simpler problems involving Bessel's functions, such for instance as are taken up in Byerly's text book on *Fourier's Series*, are usually omitted on the ground that the method of treatment is obvious. Here, as in the case of the earlier part of the book, the great diversity of subjects treated, borrowed from German as well as English writers, is an excellent feature.

CHAPTER X: *Vibrations of Membranes.* The free vibrations of circular membranes are here very briefly treated. Many interesting questions connected more or less closely with this subject and involving the use of Bessel's functions are not even mentioned, owing doubtless to the excellent treatment in Lord Rayleigh's *Theory of Sound*. It may be mentioned that formula (21) at the close of the chapter reduces at once by means of the relation

$$J_n(x)\,Y'_n(x) - Y_n(x)\,J'_n(x) = \frac{1}{x},$$

to the form:

$$L = \frac{\pi}{2\varkappa^2[J_n(\varkappa a)]^2}\left(\frac{1}{[Y_n(\varkappa b)]^2} - \frac{1}{[Y_n(\varkappa a)]^2}\right).$$

CHAPTER XI: *Hydrodynamics*, opens with a tripos problem concerning vortex motion in which the lines of flow lie in planes passing through an axis. It does not look as though the authors had noticed that the transcendental equation (10) for determining n has no real roots. In order to avoid imaginaries, the functions J_1, Y_1, and cosh should have been used instead of I_1, K_1 and cos. There follows a sketch of an investigation by Lord Kelvin on the small oscillations of a cylindrical vortex about a state of steady motion, and on waves in a cylindrical tank. The chapter closes with a treatment of the two dimensional motion of a viscous liquid which is of peculiar interest as introducing for the first time in the book Bessel functions whose arguments are neither real nor pure imaginary, but of the form $a\sqrt{i}$ where a is real.

CHAPTER XII: *Steady Flow of Electricity or of Heat in Uniform Isotropic Media.* While the problems considered in the preceding chapter led to developments in series of the sort considered in chapter VI, those considered in the present chapter lead to developments not in series, but in definite integrals. The first problem taken up, while simple, is typical of the kind of question treated in this chapter, and it will be worth while for us to consider it in some detail. The problem is, to find the electrostatic potential due to a circular conducting disk charged with electricity. The origin is taken at the centre of the disk and cylinder coördinates z, r, φ are used, the axis of z being perpendicular to the plane of the disk. As the potential is symmetrical with regard to the plane of the disk we need consider only that half of space in which z is positive. The method pursued by the authors is to write down the expression:

$$V=\frac{2c}{\pi}\int_0^\infty e^{-\lambda z}\sin(\lambda r_1)J_0(\lambda r)\frac{d\lambda}{\lambda},$$

c being the potential on the disk and r_1 the radius of the disk, and then to verify that this is the desired potential.* This method, while perfectly conclusive, is not satisfactory to the reader as it does not put him in a position to solve a similar problem for himself. We may, however, proceed as follows: From the known values of the surface density at points of the plate we have when $z=+0$ and $r<r_1$:

$$\frac{\partial V}{\partial z}=-\frac{2c}{\pi}\frac{1}{\sqrt{r_1^2-r^2}}.$$

When $z=0$ and $r>r_1$, we have from symmetry: $\frac{\partial V}{\partial z}=0$. We need then merely to find a solution of Laplace's equation which satisfies these conditions and vanishes in the proper way at infinity. Since we have circular symmetry about the axis of z we shall try to build up our potential from solutions of the form: $e^{-\lambda z} J_0(\lambda r)$. Since there is no reason to restrict the value of λ here we shall naturally form from these particular solutions not a series but an integral:

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* This verification is unnecessarily long; formula (13) *p.* 126 and the corresponding formula when $r>r_1$ being all that is necessary.

$$V = \int_0^\infty a(\lambda)\, e^{-\lambda z} J_0(\lambda r)\, d\lambda.$$

From this we get:

$$\frac{\partial V}{\partial z} = -\int_0^\infty \lambda a(\lambda)\, e^{-\lambda z} J_0(\lambda r)\, d\lambda.$$

We have then merely to determine the coefficient $a(\lambda)$ in such a way that:

$$\int_0^\infty \lambda a(\lambda)\, J_0(\lambda r)\, d\lambda = \begin{cases} \dfrac{2c}{\pi} \dfrac{1}{\sqrt{r_1^2 - r^2}} & \text{when } r < r_1, \\ 0 & \text{when } r > r_1. \end{cases}$$

This can be done at once by means of formula (166) p. 80, which gives:

$$a(\lambda) = \frac{2c}{\pi} \int_0^{r_1} \frac{\rho}{\sqrt{r_1^2 - \rho^2}} J_0(\lambda\rho)\, d\rho.$$

This definite integral can be evaluated (cf. Sonine, Math. Ann. Bd. 16 p. 36 bottom) and gives:

$$a(\lambda) = \frac{2c}{\pi} \frac{\sin(\lambda r_1)}{\lambda}.$$

Thus we are led to the value of V given above.

Further problems of a similar, though more complicated nature are treated in the remainder of this chapter which follows closely a paper by H. Weber. Many readers will doubtless wish that the authors had seen fit to devote a little more space to Riemann's treatment of Nobili's rings, as the original is far from easy reading.

CHAPTER XIII: *Propagation of Electromagnetic Waves along Wires.*—As a starting point for this recently developed and already very important subject the equations of the electromagnetic field are first given in general form, and then so specialized as to fit the case in which the waves are guided by a straight wire. x being the distance measured along the wire, ρ the distance from the wire, and t the time, the cases considered are those in which the forces are the real parts of expressions of the form:

$$f(\rho)\, e^{(mx - nt)i}$$

n being real, but m, in general, complex. The Bessel's functions which are thus introduced have complex arguments. The methods of work are in the main, those of J. J. Thomson and Hertz. No questions involving developments in series are taken up.

CHAPTER XIV : *Diffraction of Light* contains rather elaborate investigations, chiefly due to Lommel, of the diffraction produced by a small circular opening in a screen, in the course of which interesting numerical results accompanied by diagrams are given. Bessel's functions are here introduced in a way different from anything that has gone before (except the treatment of Kepler's equation), occurring directly as definite integrals and not as elsewhere as the solution of a differential equation. Some extensive analytical investigations find place in this chapter which might better, we think, have been placed in the earlier theoretical part of the book.

CHAPTER XV: *Miscellaneous Applications*, though short, is of decided interest. We choose for special mention from among the half dozen questions here considered the following beautiful problem due to Greenhill: a circular elastic cylinder of small cross section, (*e. g.*, a knitting needle) is held in a vertical position with its lower end clamped and upper end free ; to find the greatest length consistent with stability.

There follow two notes, one on Bessel's functions of the second kind, and one on the formula due to J. McMahon for the calculation of the roots of Bessel's functions. The next fifteen pages are devoted to examples to be worked out by the reader. Among these pure mathematics predominates but physical problems also occur in considerable number.

Last but not least come the numerical tables. Of these the first three are due in their present form to Meissel, while the other three are taken from the reports of the British Association. Table I is a twelve place table of the values of $J_0(x)$ and $J_1(x)$ from $x=0$ to $x=15.5$ at intervals of 0.01. This is merely an enlargement of Bessel's original table. Table II which is here published for the first time gives for positive integral values of n and x the values of $J_n(x)$ from $x=1$ to $x=24$ to eighteen decimal places. Table III gives to sixteen decimal places the first fifty roots of $J_1(x)=0$ with the corresponding values of $J_0(x)$. Table IV gives the complex values of $J_0(x\sqrt{i})$ for real values of x from 0 to 6 at intervals of 0.2. The values are given to nine decimal places. Table V gives the values of $I_1(x)=-i\,J_1(ix)$ from $x=0$ to $x=5.1$, at intervals of 0.01 to nine decimal places. Table VI gives, chiefly to twelve significant figures, the values of $I_n(x)$ for integral values of n from 0 to 11 and for values of x at intervals of 0.2 from 0 to 6. This collection of tables will be highly appreciated by all who have to use Bessel's functions in numerical work. A table similar to table III but giving the roots of the equation $J_0(x)=0$ and the corresponding values of $J_1(x)$ would be a welcome addition in a future edition. Such a table would be useful in the numerous problems involving the development of unity in the interval from zero to a in a series of the form $\sum_{s=0}^{s=\infty} A_s J_0(\lambda_s r)$ where λ_s is the s^{th} root of the equation $J_0(\lambda a)=0$.

A short bibliography which though confessedly incomplete will be found useful, and a drawing of the curves $y=J_0(x)$ and $y=J_1(x)$ close the volume.

Serious misprints seem to be rare. On page 14, however, there is one which deserves mention as it occurs in an important formula. The last term in formula (31) reads $-\sum_1^n \frac{1}{2s}$. The minus sign should be changed to plus.

MAXIME BÔCHER.

HARVARD UNIVERSITY,
March, 1896.

MODERN METHODS OF ANALYTICAL GEOMETRY.

An Introductory Account of Certain Modern Ideas and Methods in Plane Analytical Geometry. By CHARLOTTE ANGAS SCOTT, D. Sc., Girton College, Cambridge; Professor of Mathematics in Bryn Mawr College, Pennsylvania. xii+288 pp., octavo. London and New York, Macmillan and Co. $2.50.

A minor excellence of this book, for which many readers will feel truly grateful, is the fact that it is written in the English of English speaking and writing people. Private abbreviations, cabalistic marks necessitating constant reference to an elusive "list of signs," Teutonisms, and Greek logomachy in the way of "tetrastigms," etc., are agreeably absent. The parvenu "join" is flattered with recognition, but this term is now in such general use that to protest further against it will be of little avail. It is in a measure a consolation that no one is as yet permitted to "enthuse" over this acquisition to the language. "Joining line" is a letter and a syllable shorter than "intersection," the correlative term. If we say "join" shall we not say "meet" or "cross"? "Double point" also might better be written in full. But these are trivial affairs.

The book modestly designated by the author as an "introduction," and certainly well adapted to the needs of the learner, is in fact a compact scholarly work on the more accessible principles and methods of modern analytical geometry. It exhibits to a marked degree that genial breadth of treatment and conciseness which are associated only with mature scholarship and extensive and accurate information. Everything fits naturally in place, nothing is cramped or forced out of its natural relations. In the limited space of less than three hundred pages the great ideas of geometry, including the doctrine of coördinates, projection and dualism, correspondence, and the absolute, are clearly and skillfully developed, and illustrated by application to an ample list of well-chosen examples, a sufficient number of the latter being reserved as an exercise for the reader. We notice with pleasure that the book is entirely free from the criticism, so often and so properly urged against many English mathematical works, that their authors exhibit no adequate conception of the distinction between a general principle and a particular example. Miss Scott is not only keenly alive to this distinction; she is also perfectly aware of the importance of choosing methods adapted to the subject, which is here primarily geometry, not algebra.

Historical and other references are only rarely given, on the assigned ground of the introductory character of the book. Some years ago a prominent French mathematician offered a different excuse for the same omission, viz., that he could not by giving references enhance the fame of the illustrious geometers who had preceded him. We apprehend that neither plea will be good in court. The beginner as well as the scholar should have every facility for acquainting himself with the history and classical works of his subject. And, as we have pointed out, in the present case "introductory" is not by any means synonymous with "elementary."

The author's mode of presentation, usually good, becomes occasionally very felicitous. At the beginning the justification of the line as an element, co-important with the point, of the plane is well stated. The infinite and imaginary elements—line at infinity, circular points, isotropic lines, etc., are skillfully managed. The chapter on the Absolute is an admirably lucid and elementary exposition. The author aims throughout at generality and elasticity of treatment. Her success in this respect is very pronounced, and is in fact one of the most acceptable features of the book.

The book covers of course to a large extent the same field as the treatises of Salmon to which the author very appreciatively refers, and to which the reader is directed for the study of many aspects of the subject. No mention is however made of the German editions of Salmon's works, which are in some cases greatly superior to the originals. There is ample room for new and modern books in various departments of geometry, and we will trust that Miss Scott's very successful present venture into this field will not be her last.

Pure geometry travels flying light, without baggage, and lives on the country. The analytical arm of the service carries a heavy, but nowadays perfectly mobile, siege train with many supply depots. And first of all the accoutrements in the way of coördinates must be well appointed. A coördinate is any quantity that helps to determine an element, and an element is any geometric form. The simplest elements of plane geometry are the point and the line. On each of these a geometry can be constructed; and these two geometries are identical in substance and differ only in guise. How many coördinates a given element may have is a matter of taste. Cartesian geometry, content with bare necessaries, considers only the point as element and assigns to this two coördinates. With three coördinates for the point, (or line) however, the algebraic evolutions become much more symmetric, inasmuch as all equations are then homogeneous. At the the same time, and this is the real advantage, the whole coördinate system becomes projective, and accompanies the corps of figures to which it is attached in their passage from one plane to another. The three coördinates of point or line are referred to the sides and vertices of a triangle of reference, most conveniently the same triangle in both cases. In the Cartesian geometry, as viewed in this aspect, the coördinate axes are two sides of this triangle of reference, while the third side is the conventional "line at infinity." The latter is a line, geometrically because its projection is a line, and analytically because the equations of all lines infinitely removed tend to the same form. Since it is true that every quadrangle can be projected into every other, it might seem desirable to employ four coördinates instead of three. In fact, however, one vertex of the quadrangle is already in demand as an adjustment—unit point, *vide infra*—for the coördinate system attached to the other three.

If a plane figure is projected into a new plane, the triangle of reference of the original plane becomes a triangle which may be used as a triangle of reference in the new plane. This being the case the equations of the two figures are identical. If, however, it is desired to change the triangle of reference in a plane, this is effected by an integral homogeneous linear transformation of coördinates with non-vanishing discriminant. The equations of transformation admit of a second interpretation, viz.: the triangle of reference may be regarded as fixed while the plane undergoes distortion. In the latter case any figure is distorted into a projective figure. The discussion of the relation between projection and linear transformation is elaborated in a special chapter.

Since two coördinates are sufficient to determine a point or line, the three coördinates actually employed in each case are connected by an identity. The latter is linear for point coördinates, but, curiously enough, for line coördinates it is quadratic. In this violation of the principle of dualism we have an epitome of the one-sided character of our system of metric, leading to no little inconvenience until it finally comes to be recognized as a one-sided limiting form of a more general conception in which the principle of dualism retains its validity in all cases, and at the same time all properties are projective. Properties are then still divided into two classes according as they belong to a figure in itself or express its relations to the absolute conic. Until this step is taken, however, it follows from the identities above mentioned that we have only one singular (infinitely distant) line, but two infinitely distant points. Closely connected with this is the fact that there is in our ordinary geometry no natural unit of length, while there is a natural unit angle, viz., the complete circumference.

A coördinate, as an algebraic quantity, may, and not only may but inevitably will, assume imaginary as well as real values, and its imaginary values are infinitely more numerous than its real ones. The common conception of geometry, therefore, requires to be idealized by the introduction of imaginary *geometric* elements. Writers on analytical geometry do not ordinarily concern themselves with the development of this idea and the author follows the usual practice. Any extensive treatment of the imaginary may well be left to special treatises, but some of the more elementary notions might properly have been mentioned here. For example, what is called the equation of a plane curve really represents a surface of which the curve is only a section; two such surfaces intersect in a finite number of distinct points; isolated points of a curve are happily explained in this connection. The nature of the continuity of the surface is readily ascertained, and the theories of deficiency and of birational transformation are intimately related to this subject. But this is already going rather far,—beyond the limit that the author has set for the work.

The great unifying and fertilizing principle of geometry is the doctrine of projection, or more generally of correspondence. The great advances which analytic geometry has made during the present century are due entirely to the adoption of projective ideas and methods. With this has gone a recasting of the entire notation of the science into projective form by the aid of anharmonic ratio. And thus finally all essential distinction between synthetic and analytic geometry has long since disappeared. Still our modern works on analytic geometry are very distinctively analytic, and writers on pure geometry, almost without exception, avoid carefully any explicit mention of coördinates. But it is an easy prediction that this fictitious boundary line will not long be observed. The author of the *Introduction* gives liberal recognition to the methods of pure geometry; but we could wish that she had gone still further in this direction. For example an alternative, purely geometrical, proof of the characteristic property of the complete quadrangle and quadrilateral (p. 43) might well have been given. We miss, too, the many easy geometric deductions from the notion of projective ranges and pencils, including ranges on a conic and tangents of a conic, especially in the latter case the geometric proof of Pascal's and Brianchon's theorems, which in point of conciseness and exact adaptation to the subject are simply perfection. One matter in particular is so important and so pertinent to *analytic* geometry, that we cannot but regard its omission as unfortunate. It is well known that the anharmonic ratio is the simplest *invariant*. It is also true that the anharmonic ratio is a *universal coördinate*, all other coördinates being merely disguised forms of anharmonic ratios. In fact, the anharmonic ratio is referred to three elements at which it takes the values ∞, 1, 0, respectively, and if any coördinate x of an element P takes these three values at the elements, A, B, C, then $x = (ABCP)$. For example, the Cartesian coördinate x of a point on a line is an anharmonic ratio referred to the point at infinity A, the origin C and the unit point B, CB

being the arbitrary unit of length. Again the ratios of the trimetric coördinates of a point, P, referred to a triangle, ABC are

$$\frac{x_1}{x_2}=(C-ADBP),\ \frac{x_2}{x_3}=(A-BDCP),\ \frac{x_3}{x_1}=(B-CDAP),$$

when D is the unit point, *i. e.*, the point for which $x_1 : x_2 : x_3=1$. On this principle depends the possibility of making analytic geometry thoroughly projective, the coördinates themselves admitting of projective definition.

Geometric properties are primarily classified as projective and non-projective. Beside these the terms "descriptive" and "metrical" are in common use and are given prominence by the author. According to Cayley (Salmon's Higher Plane Curves, Chapter I) properties are descriptive if they have no reference to the line at infinity or the circular points; otherwise they are metrical. In this use "descriptive" is identical with "projective" and "metrical" with "non-projective." The author's definition of "descriptive" is the analytical equivalent of Cayley's, but this definition is not strictly adhered to inasmuch as properties expressed in terms of anharmonic ratios are afterward described as metrical (p. 147). It seems to us that the term "descriptive" which is commonly employed in conjunction with "projective" and non-projective, as a species of *tertium quid*, might well be reserved to designate the technical art to which it was originally applied. "Metrical" has an intrinsic value, and in the sense of "involving measurement" could hardly be dispensed with. That its use is however liable to be attended with some ambiguity is evidenced by the statement (p. 147). "Theorems stated with reference to cross-ratio are metric theorems in projective form," which to our mind rather gains in force if the words "metric" and "projective" are interchanged.

In pure geometry projection is a matter complete in itself, but in analytic geometry it has its counterpart in the theory of linear transformation. The latter subject may be fairly said to lie at the very center of modern mathematics. From the formal standpoint projective geometry is merely an interpretation in space of linear transformation, an interpretation so interesting and valuable that when, with an increased number of variables in the equations, ordinary space no longer suffices for the purpose, no hesitation is felt in positing a hyper-space with any number of dimensions. In fact the study of geometry has, for many, its chief value in the fact that it furnishes an abundance of simple and current illustrations of principles, whose abstract foundation belongs to other more difficult mathematical sciences. For example, as the author very appropriately points out, all the collineations of a given space form a *group*. This group has a variety of *subgroups*, and it has also its peculiar *invariants* and *covariants*.

The chapter on Projection and Linear Transformation is followed by one on Correspondence, in which the general notion of correspondence is explained and applied to the general (1, 1) quadric correspondence, to reciprocation, and briefly to the birational transformation of a curve into itself. Following this is the chapter on the Absolute, and finally a very short but well-written introduction to Invariants and Covariants. Descriptive and metrical properties, and the theory of anharmonic ratio and involutions are treated at length, but compactly in the earlier chapters. There are also separate chapters on Dualism and on Unicursal Curves.

The book will be found to gain continually in interest on repeated reading, and this is due especially to its eminently suggestive character. The author succeeds admirably in conveying to the reader, if he be fit, an accurate knowledge and command of general principles. We know of no introductory work which is better adapted in this particular for the use of those who desire not merely to learn but also to master geometry.

F. N. COLE.

COLUMBIA UNIVERSITY,
March, 1896.

ON A REMARKABLE COVARIANT OF A SYSTEM OF QUANTICS.

BY PROFESSOR H. B. NEWSON.

I wish to call attention in this paper to a remarkable covariant of a system of n quantics in n homogeneous variables. This covariant although not entirely new has not received the attention which it deserves. While the exposition here given is limited to the case of three quantics in three homogeneous variables, it will be seen that the methods are applicable to a system of n quantics in n variables. This covariant is here approached from the geometric point of view and the language used is that of the theory of higher plane curves. I shall attempt only to sketch the work in outline and shall omit all details as unsuitable to the place and object of this paper.

The covariant in question bears much the same relation to the Jacobian of the system of quantics as the Steinerian of a single quantic bears to the Hessian. In order therefore to get the most advantageous starting point for the discussion, we shall begin by restating two fundamental properties of the Jacobian.

Let there be given three curves, U, V and W, whose degrees are respectively m, n, and p. The Jacobian of these three curves, whose equation is obtained by equating to zero the functional determinant of U, V and W, has two well known polar properties as follows:

(1). *The Jacobian of* U, V *and* W *is the locus of all points common to the first polars of a point* (x', y', z') *with respect to* U, V *and* W.

(2). *The Jacobian of* U, V *and* W *is also the locus of the point* (x', y', z'), *whose polar lines with respect to* U, V *and* W, *meet in a point.*

The equation of the Jacobian is obtained in two different ways, each of which leads to one of the properties in question. The first polars of (x', y', z') with respect to U, V and W, are given by the following equations, using Salmon's notation (see Higher Plane Curves, Art. 61).

$$\begin{aligned} x'U_1+y'U_2+z'U_3&=0, \\ x'V_1+y'V_2+z'V_3&=0, \\ x'W_1+y'W_2+z'W_3&=0. \end{aligned} \tag{1}$$

When these three equations are simultaneous their resultant vanishes, and we have the equation of the Jacobian and the proof of property (1).

The polar lines of (x', y', z') with respect to U, V and W are given by the equations

$$
(2)\qquad
\begin{aligned}
xU_1'+yU_2'+zU_3'&=0,\\
xV_1'+yV_2'+zV_3'&=0,\\
xW_1'+yW_2'+zW_3'&=0.
\end{aligned}
$$

When these three lines meet in a point the resultant of their equations vanishes, and we have again the equation of the Jacobian and the proof of the second property.

The Jacobian is a curve of order $(m+n+p-3)$.

But instead of confining our attention to the Jacobian we might also consider the locus of the point (x', y', z') whose first polars meet in a point. The equation of this locus is obtained by eliminating x, y and z from equations (1). It is easy to see that we obtain the same equation by eliminating x', y' and z' from equations (2), except that x, y, z are interchanged with x', y' and z'.

I propose to name this locus provisionally the Cremonian of U, V and W; I say provisionally because I do not know by whom the curve was first studied. It is specifically mentioned in Cremona's *Introduzione ad una teoria geometrica delle curve piane*, Art. 98. See also Hagen's *Synopsis der Hoeheren Mathematik*, vol. 2, page 201. In Higher Plane Curves, Art. 401, Salmon mentions this locus which is here called the Cremonian, but says: "We shall confine ourselves to the consideration of the case when the three curves are first polars of a given curve, in which case the Jacobian in the Hessian of that curve, and the other locus now mentioned is its Steinerian." It seems to me that the general case is too important to be dismissed in this way; in fact more important than the special case which is frequently mentioned.

The Cremonian has two important geometric properties corresponding to the two methods by which its equation may be obtained.

(1). *The Cremonian of U, V and W is the locus of the point (x', y', z') whose first polars with respect to U, V, and W have a common point; the locus of these common points is of course the Jacobian.*

(2). *The Cremonian of U, V, and W is also the locus of (x, y, z) the point of intersection of the polar lines of (x', y', z'), with respect to U, V and W; i. e., it is the locus of the point of intersection of the polar lines of the points on the Jacobian.*

If we select some point as C on the Cremonian, its first polars intersect in a point J on the Jacobian; the polar lines of the point J meet in a point on the Cremonian. It is not difficult to see that this last point must be the point C. Thus there are two relations connecting the points C and J, which may well be called corresponding points on the two curves. All the properties of the Jacobian and the Cremonian so far mentioned may be summed up in the following comprehensive theorem :

If the first polars of a point C with respect to three curves U, V and W meet in a point J; then the polar lines of the point J with respect to the same three curves meet in the point C; and vice versa. The locus of all points C is the Cremonian of the three curves, and the locus of all points J is the Jacobian of the three curves.

When we consider the process of eliminating x, y and z from the three equations (1), it is easy to see that the Cremonian is of order $(mn+mp+np-2m-2n-2p+3)$. And since the two curves have a one to one correspondence, we infer that the Cremonian and the Jacobian have the same deficiency.

The lines joining corresponding points on the Jacobian and Cremonian, envelope a curve for which I have no better name than the Hyper-Cayleyan ; because it is generated in a manner similar to the method of generating the Cayleyan. It is evident that the Hyper-Cayleyan has a one to one correspondence both with the Cremonian and the Jacobian, and has therefore the same deficiency. The class of this Hyper-Cayleyan is, we know, equal to the sum of the orders of the Jacobian and the Cremonian, *i. e.*, it is given by the number $(mn+mp+np-m-n-p)$.

It is not a difficult task to prepare a table of the Pleuckerian characteristics of the three curves, the Jacobian, the Cremonian and the Hyper-Cayleyan, similar to that for the Hessian, Steinerian and the Cayleyan given in Clebsch-Lindemann's *Vorlesungen ueber Geometrie*, vol. I., page 368. This table is also given in Hagen's *Synopsis der Hoeheren Mathematik*, vol. II., page 203.

In the most general case we suppose that the three curves, U, V, W, have no double points, are of different degrees, m, n, p, and have no points or points common to all three. A study of the special cases brings to light a large number of interesting theories. Thus we may consider the cases where one or more of the curves have double points, cusps or multiple points: also the cases where two and then three of the curves have points in common, which may be ordinary points on the curve or multiple points of various orders.

When U, V, W are all of the same degree, we know that the Jacobian of the three curves is the locus of the double points on all the curves of the system $U+kV+lW=0$. When U, V, W are all of the same degree and at the same time are first polars of three points with respect to a curve C of degree $(m+1)$ or n; then the Jacobian of U, V, W becomes the Hessian of C, the Cremonian of U, V, W becomes the Steinerian of C, and the Hyper-Cayleyan of U, V, W becomes the Cayleyan of C.

It is not difficult to extend the conception of the Cremonian here developed to spaces of higher dimensions. In three dimensions we must have given four surfaces, U, V, W, T; the Jacobian and Cremonian are then defined and their equations obtained in a manner analogous to that employed in two dimensions. The lines joining the corresponding points on the Jacobian and Cremonian surfaces form a system of ∞^2 lines in space. I have not examined the properties of this system of lines.

In a paper published in the *Kansas University Quarterly* vol. 3., No. 2, entitled *On the Hessian, Jacobian, Steinerian, etc., in Geometry of One Dimension*, the writer has developed some of the properties of the Cremonian of two binary quantics. In that paper the name Cremonian was not used; but the covariant in question was designated by the symbol $M(VW)$.

So far as my knowledge goes there has been no systematic and detailed study of the properties of the covariants, which are here called the Cremonian and Hyper-Cayleyan. It seems to me that they are worthy of serious investigation, and it is to be hoped that some one will undertake the task.

KANSAS STATE UNIVERSITY,
February 17, 1896.

NOTES.

A REGULAR meeting of the AMERICAN MATHEMATICAL SOCIETY was held in New York, Saturday afternoon, March 28th, at three o'clock, the President, Dr. HILL, in the chair. There were fifteen members present. Two nominations for membership were received. The following papers were read:

(1) Professor H. B. NEWSON: "On a remarkable covariant of a system of n quantics."

(2) Professor E. W. BROWN: "On the particular integrals of linear differential equations, and their application to the lunar theory."

In the absence of Professor NEWSON, his paper was read by the Secretary.

THE Secretary of the Society has been directed by the Council to prepare a new List of Members for early publication, and has accordingly issued a circular asking for necessary information from the members of the Society. Those members who have not already responded to this circular are requested to do so at once.

A SUMMER meeting of the Society will be held at Buffalo on Monday and Tuesday, August 31st and September 1st, in affiliation with the American Association for the Advancement of Science. Members intending to contribute papers or to be present are requested to notify the Secretary.

FACULTÉ DES SCIENCES. During the second semester of the current academic year, the following mathematical courses are offered at the Sorbonne, in Paris:—Professor Hermite: Theory of Eulerian integrals and the theory of elliptic functions;—Professor Picard: Theory of twisted curves and of algebraic surfaces;—Professor Appell: General laws of the motions of systems, analytical mechanics, hydrodynamics;—Professor Wolf: Astronomical subjects included in the programme of the *Licence*;—Professor Poincaré: Optics;—Professor Boussinesq: Theory of oscillatory waves (rise and fall of the sea; waves produced on the surface of still water by the immersion of a solid, or by superficial impulsion like that of an air current, etc.). The following supplementary course is also given: M. Painlevé: Differential and integral calculus. Mathematical conferences are conducted by Messrs. Painlevé, Puiseux, Raffy, Andoyer and Blutel.

HARVARD UNIVERSITY. During the year 1896–97, Professor Asaph Hall (U. S. Navy) will give a course in Celestial Mechanics. The following advanced mathematical courses are also offered:— By Professor J. M. Peirce: Algebraic Plane Curves; Quaternions (second course); Special Developments of the Calculus of Quaternions †.—By Professor Byerly; Differential and Integral Calculus (second course); Analytic Mechanics (with Mr. Whittemore); Rigid Dynamics †.—By Professors Byerly and B. O. Pierce: Fourier's Series, Spherical Harmonics and Potential Function.—By Professor B. O. Pierce: Theory of Surfaces †.—By Professor Osgood: Infinite Series and Products †; Elliptic Functions.—By Professor Bôcher: Theory of Equations and Invariants †; Modern Geometry; Theory of Numbers †; Theory of Functions (first course); Bessel's Functions †.

The above courses will each consist of three lectures a week throughout the entire academic year except those marked † which consist of at least one-half that number of lectures. The following reading courses are also offered:—By Professor J. M. Peirce: Studies in Mathematical Literature.—By Professor B. O. Peirce: Elasticity.—By Professor Osgood: Klein's Ikosaeder.

In addition to the above the mathematical department also offers instruction in the following elementary subjects: Solid Geometry, Algebra, Logarithms, Plane and Spherical Trigonometry, Analytic Geometry of two and three dimensions, Differential and Integral Calculus, Mechanics.

THE UNIVERSITY OF CHICAGO. The mathematical courses to be offered during the summer quarter of 1896 have already been announced in the BULLETIN (p. 183).

During the three quarters (*a*, *w*, *s*) of the academic year 1896–97 the following advanced mathematical courses (four or five hours weekly) will be offered: By Professor Moore, Seminar devoted to research work and Kronecker's arithmetical theory of algebraical quantities (*a*, *w*), Theory of groups (*w*), Projective geometry (*a*); by Professor Bolza, Galois theory of algebraic equations (*s*), Solid analytics (*s*); by Associate Professor Maschke, Modern analytic geometry (*a*), Higher plane curves (*w*), Algebraic surfaces (*s*), Theory of functions of a complex variable (*s*), Advanced integral calculus (*a*, *w*); by Dr. Young, Theory of numbers (*w*); by Dr. Boyd, Differential-geometry of surfaces (*a*); by Dr. Hancock, Calculus of variations (*s*), Theory of equations (*a*, *w*); by Dr. Laves, Analytic me-

chanics (a, w); by Mr. Slaught, Differential equations (s).
The Mathematical Club meets fortnightly.

THE Council of The AMERICAN MATHEMATICAL SOCIETY has given its approval to a proposition for holding a colloquium in connection with the summer meeting at Buffalo next August. It is thought by many that the benefits arising from the summer meeting will be greatly increased, if those attending have an opportunity of hearing brief accounts of the progress made in different branches of the higher mathematics. It is intended to have two or three short courses of lectures by eminent mathematicians, every lecture being followed by an informal discussion. Any arrangements that may be made will be announced in a later number of the BULLETIN.

A NEW American mathematical journal is soon to appear. It will be edited by Professor W. E. Story of Clark University, Worcester, Mass., and will be issued bimonthly. Each number, it is said, will contain, in addition to papers devoted to mathematical research, notices of the transactions of learned societies and the titles of recent mathematical books and articles. Each number will consist of about ninety-six large octavo pages. The subscription price will be $5.00 per year.

WE take pleasure in announcing the appearance of the volume of *Mathematical Papers read at the International Mathematical Congress held in connection with the Columbian Exposition, Chicago*, 1893. The publishers are Macmillan and Co. The price of the volume is $4.00.

A Mathematical work of unusual character is announced for July next, by John Wiley and Sons. It is entitled *Higher Mathematics for Engineering Colleges*, and is edited by Professor Mansfield Merriman and Professor R. S. Woodward. The volume is intended for advanced students in engineering colleges and other institutions, and is designed to supplement their elementary mathematics by a knowledge of the elements of such branches of modern higher mathematics as admit of practical application. In addition to chapters by the editors on the *Solution of Equations* and *Probabilities and Theory of Errors*, the work will contain the following chapters: Professor W. E. Byerly, Harvard University, *Harmonic Functions*; Professor Thomas S. Fiske, Columbia University, *Functions of a Complex Variable*; Professor G. B. Halsted, University of Texas, *Projective Geometry*; Professor E. W. Hyde, University of Cincinnati, *Point Analysis and Aus-*

dehnungslehre; Professor W. W. Johnson, U. S. Naval Academy, *Differential Equations*; Professor A. Macfarlane, Lehigh University, *Vector Analysis and Quaternions*; Professor James McMahon, Cornell University, *Hyperbolic Functions*; Professor F. Morley, Haverford College, *Elliptic Integrals and Functions*; Professor D. E. Smith, Michigan Normal School, *History of Modern Mathematics*; Professor L. G. Weld, University of Iowa, *Determinants*.

ACCORDING to *Nature* the French Association for the Advancement of Science held a very successful meeting at Tunis beginning April 1st. More than four hundred members were present. The principal streets of Tunis were elaborately decorated in honor of the occasion, and, as hotel accommodation was limited, the Lyceé Carnot was placed at the disposal of visitors. The association met in Algiers fifteen years ago, so that this was the second meeting on African soil. Next year's meeting will take place at St. Etienne.

WE learn from the *Revue génerale des Sciences pures et appliquées* that Professor Mittag-Leffler, founder of the *Acta Mathematica*, completed last March his fiftieth year. On that occasion a number of his admirers presented him with a congratulatory address written in four languages, German, English, French and Italian, and his own portrait done in oil by the Finnish painter Edelfelt.

AT the April meeting of the National Academy of Sciencs Professor R. S. Woodward, of the department of mechanics in Columbia University, was elected to membership.

NEW PUBLICATIONS.

I. HIGHER MATHEMATICS.

Albert (G.). Die Platonische Zahl und einige Conjecturen zu Platon sowie zu Lukrez. Wien, 1896. 8vo. 1 plate. Mk. 1.00.

Borel (E.). See **Niewenglowski** (B.).

Caronnet (T.). Problèmes de licence ès sciences mathématiques donnés à la Sorbonne. Avec les solutions développées. (Année 1895, sessions de juillet et de novembre, élèves libres et élèves de l'École normale.) Paris, Nony, 1895. 8vo. Fr. 1.00

Carvallo (E.). Méthode pratique pour la résolution numérique complète des équations algébriques ou transcendantes. Paris, Nony, 1896. 4to. Fr. 1.50

Edwards (J.). An elementary treatise on the differential calculus, with applications and numerous examples. 3d edition, revised and enlarged. London, Macmillan, 1896. 8vo. 546 pp. 14s.

Galopin-Schaub (C.). Premières notions du calcul des quaternions. Bâle, 1896. 12mo. 64 pp. Mk. 1.50

Goursat (E.). Leçons sur l'intégration des équations aux dérivées partielles du second ordre à deux variables indépendantes. Vol. I: Problème de Cauchy; caractéristiques; intégrales intermédiaires. Paris, Hermann, 1896. 8vo. 8 and 226 pp. Fr. 7.50

Grassmann (H.). Gesammelte mathematische und physikalische Werke. Band I, Theil 2: Die Ausdehnungslehre von 1862; in Gemeinschaft mit H. Grassmann dem Jüngeren herausgegeben von F. Engel. Leipzig, Teubner, 1896. 8vo. 8 and 512 pp. Mk. 16.00

Haas (A.). Beiträge zur graphischen Darstellung der höheren Plankurven. [Progr.] Stuttgart, 1895. 4to. 2 and 52 pp.

Koenigsberger (L.). Hermann von Helmholtz's Untersuchungen über die Grundlagen der Mathematik und Mechanik. Mit einem Bildniss Hermann von Helmholtz's. Leipzig, Teubner, 1896. 8vo. 6 and 58 pp. Mk. 2.40

Martin (Artemas.). About cube numbers whose sum is a cube number. [*Mathematical Magazine*, Vol. 2, pp. 153–160 and 185–190.] 4to.

——— About biquadrate numbers whose sum is a biquadrate. [*Mathematical Magazine*, Vol. 2, pp. 173–184.] 4to.

Maschke (H.). Ueber die Darstellung endlicher Gruppen durch Cayley'sche Farbendiagramme. [*Nachrichten der K. Gesellschaft der Wissenschaften zur Göttingen*, math.-phys. Klasse, 1896, Heft 1.] 8vo. 6 pp.

Mathematical Questions and Solutions from the *Educational Times*. Vol. 64. Edited by W. J. C. Miller. London, Hodgson, 1896. 8vo. 128 pp. Boards. 6s. 6d.

Milinowski (A.). Elementar-synthetische Geometrie der Kegelschnitte. Mit einem Anhange über die gleichseitige Hyperbel. 2te Ausgabe (reduced in price). Leipzig, Teubner, 1896. 8vo. 12 and 412 pp., 10 and 135 pp. Mk. 4.00

Miller (G. A.). Sur les groupes de substitutions. [*Comptes rendus*, Vol. 122, pp. 370–372]. 4to.

Miller (W. J. C.). See Mathematical Questions and Solutions.

Niewenglowski (B.). Cours de géométrie analytique. (En 3 volumes.) Vol. III. : Géométrie dans l'espace, avec une note sur les transformations en géométrie, par E. Borel. Paris, Gauthier-Villars, 1896. 8vo. 1er fascicule : pp. 1–336. Complete in two fascicules. Fr. 13.00

Porchon (P.). Compléments d'algèbre et de géométrie : applications de l'algèbre à la géométrie, sections coniques ; rédigés conformément aux derniers programmes officiels pour la classe de première (sciences). Paris, Alcan, 1894. 16mo. 204 pp.

Schlegel (V.) Die Grassmann'sche Ausdehnungslehre. Ein Beitrag zur Geschichte der Mathematik in den letzten fünfzig Jahren. Schluss. (*Zeitschrift für Mathematik und Physik*, Vol. 41 (1896), historisch-literarische Abtheilung, pp. 41-59.) 8vo.

Segre (C.). Intorno ad un carattere delle superficie e delle varietà superiori algebriche. [*Accademia reale delle scienze di Torino*, Anno 1895–96.] Torino, Clausen, 1896. 8vo. 20 pp.

II. ELEMENTARY MATHEMATICS.

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OF THE

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EDITED BY
THOMAS S. FISKE, ALEXANDER ZIWET, FRANK MORLEY.

2ND SERIES, VOL. II. No. 9.

JUNE, 1896.

NEW YORK:
PUBLISHED FOR THE SOCIETY
BY MACMILLAN & CO.
1896.

Entered at the Post-office at New York, N. Y., as second-class matter.
PUBLISHED MONTHLY, EXCEPT AUGUST AND SEPTEMBER.

Whole No. 49. $5.00 a ye

THE MOTIONS OF THE ATMOSPHERE AND ESPECIALLY ITS WAVES.*

Address delivered before the Meteorological Section of the Association of German Naturalists and Physicians at the Annual Meeting held in Vienna, September 25, 1894.

BY DR. E. HERRMANN.

The inadmissibility of those views according to which the motions of the atmosphere consist in the development of independent cyclones and anti-cyclones is, of late years, more and more plainly recognized. This conclusion has been arrived at, not so much through a severe criticism of the fundamental basis upon which these erroneous views had been established, as by the power of the facts that resisted introduction into this artificial system.

It is now nearly ten years since the theory of cyclones began to totter, and especially under the influence of a memoir published by Hann.† Following soon upon this the idea was developed by von Helmholtz ‡ (*Berlin Sitzungsb.*, 1888), "that in the atmosphere, by continuously acting forces, the formation of surfaces of discontinuity is possible, and that the anticyclonic motion of the lower strata and the extensive, gradually increasing cyclone of the upper strata that are to be expected around the north and south poles, resolve themselves into a great number of *irregularly* progressing cyclones and anticyclones with an excess of the former." Von Helmholtz further says§ (*Berlin Sitzungsb.*, 1889): "if as in the case of the earth, the lower stratum is the denser, then it can be shown that the disturbances must at first proceed as do the waves of water raised by the wind." In this way the origin of cyclones and anticyclones is traced back to the general atmospheric circulation which is itself dependent on the difference in temperature between the equator and the poles; but the cyclones and anticyclones when once formed can still be considered as independent phenomena complete within themselves.

---

* Translated from *Verhandlungen der Gesellschaft Deutscher Naturforscher und Aerzte*, pp. 42–50 and 323–324, by Professor CLEVELAND ABBE.

† Comp. von Bezold, *Berliner Sitzungsberichte*, math.-nat. Kl., 1890, p. 831.

‡ Translated at pages 92, 93 of the *Mechanics of the Earth's Atmosphere.*

§ Translated at page 94 of the *Mechanics of the Earth's Atmosphere.*

The proof of this proposition was obtained by von Helmholtz by the application of the so-called theorem of the conservation of areas to a specific *portion* of the atmosphere, *i. e.*, a ring of air. But the application of this theorem in this manner is incorrect; it can only be applied to a free system of bodies (See *Met. Zeit.*, 1894, p. 114). Von Helmholtz applies this theorem to a ring of air cut out of the atmosphere; now such a ring of air does not form a free system but is subject to the conditions of its connection with the rest of the atmosphere, and the theorem of the conservation of angular momentum, or of areas, is not applicable to it. The application of this theorem to a ring-shaped portion of the atmosphere is only allowable in case the conditions within the atmosphere are of such a nature that they correspond entirely to the conditions at the boundary surface of a fluid and any other body. The conditions for the boundary surface of a fluid are: equal pressure on both sides of the boundary surface and equal values for the normal component of the velocities in the two masses that come in contact at the surface. (See Kirchhoff, *Mechanik*, 2d ed. 1877, p. 165; 3d ed. 1883, p. 164). These conditions are not fulfilled in the earth's atmosphere, which is in motion and not in static equilibrium. Therefore, that theory of atmospheric motions that depends upon their analysis into cyclones and anticyclones is deceptive and not so well established as it was believed to have been by the investigations of von Helmholtz.

Now, in order to find the point at which the prevailing views as to the motions of the atmosphere diverge from the reality it will be necessary to follow up a train of thought based upon indubitable well-established mechanical theorems. In doing this we will, as usual, begin with the consideration of an atmosphere that, without friction, surrounds the ideal figure of the earth, *i. e.*, a homogeneous ellipsoid of rotation, and whose particles are attracted by the mass of this ellipsoid, according to the general law of attraction of masses. It will be further assumed that, for a certain initial temperature, uniform throughout all its strata, the atmosphere has arrived at a uniform rate of revolution about the axis of rotation of the ellipsoid and with a velocity such that the ellipsoid represents the figure of equilibrium of an incompressible fluid.

Let this initial equilibrium of the atmosphere be now disturbed by introducing a different distribution of temperature, viz., one depending only on altitude above the ideal surface of the earth and on latitude, but uniform along any circle of latitude and every circle concentric therewith.

The *first* question now is whether a new state of equilibrium can develop, perhaps by means of a new rate of rotation that shall be different in the various strata of air vertically above each zone of latitude but be uniform in each circle concentric with and lying in the same plane as the circle of latitude.

The conditions for the equilibrium of a rotating fluid are:

$$\mu(R+r\omega^2)=\frac{\partial p}{\partial r},\qquad \mu Z=\frac{\partial p}{\partial z},$$

where the axis of z is the axis of rotation; μ is the density of the element of mass; r is the distance of the element of mass from the axis of z; ω is the angular velocity; R and Z are the components of the forces parallel to r and z; p is the elastic pressure. Let V be the potential of the attraction of the mass of the terrestrial ellipsoid; T the absolute temperature of the element of fluid mass. Let the characteristic equation of elasticity for gases be:

$$\frac{p}{T}=A\mu\ , \tag{1}$$

The equations of equilibrium can now be transformed into:

$$\frac{1}{T}\left(\frac{\partial V}{\partial r}+r\omega^2\right)=\frac{A}{p}\,\frac{\partial p}{\partial r},\qquad \frac{1}{T}\,\frac{\partial V}{\partial z}=\frac{A}{p}\,\frac{\partial p}{\partial z}\,.$$

By integrating the second equation there results:

$$A\log p=\int\frac{1}{T}\cdot\frac{\partial V}{\partial z}\,dz+F(r). \tag{2}$$

By differentiating this with respect to r, substituting the value of $\frac{A}{p}\cdot\frac{\partial p}{\partial r}$ thus obtained in the first equation and executing a partial integration we obtain:

$$r\omega^2=T\left\{\int\left(\frac{\partial V}{\partial z}\cdot\frac{\partial\frac{1}{T}}{\partial r}-\frac{\partial V}{\partial r}\cdot\frac{\partial\frac{1}{T}}{\partial z}\right)dz+F'(r)\right\}. \tag{3}$$

$F(r)$ and $F'(r)$ are determined by the surface conditions, in the present case by the circumstances existing in the highest layer of the atmosphere in which a uniform temperature and a velocity of rotation equal to that of the earth will be assumed. Thus, p, μ and ω are determined by the preceding equations. Under the fundamental assumption that no other external forces than the attraction of the mass of the ellipsoidal earth are acting on the atmosphere, it follows that the changes in the distribution of temperature that produce changes in the motions of the atmosphere can produce no changes in the total moment (M) with reference to the axis of z. Therefore, a state of equilibrium can be brought about in the above-mentioned manner only for such distributions of temperature in the atmosphere as, by means of the above equations, give values for μ and ω that shall satisfy the equation

$$(4) \qquad \int \mu . r^2 . \omega . dv = M,$$

the integration extending over the whole volume of the atmosphere. Therefore, a condition of equilibrium can only exist when the temperature T satisfies the equations (1), (2), (3) and (4). Such a condition of equilibrium cannot occur with every distribution of temperature, but, in general, there must exist movements in the direction from the equator to the pole, and inversely, besides a different rate of rotation for the different strata of air.

The *second* question now becomes this: Can a steady condition of motion occur in the atmosphere with an unvarying distribution of temperature? That is to say, under the assumed conditions, can the movement at each individual point in the atmosphere remain always the same?

Under the original assumption that the initial angular velocity is the same throughout the whole atmosphere, and that the temperature changes only with the distance from the pole, and with the altitude above the ideal surface of the earth, it must happen that whenever any steady condition does occur, the conditions must be the same at every point of any circle described about the axis of rotation. Therefore, throughout such a circle there can be no change of pressure with reference to either time or space [*i. e.* neither chronological nor geographical].

Such a distribution of pressure, however, cannot coexist with motions having a meridional component. This can be proved as follows:

Assuming that the temperature in the atmosphere depends only on the latitude and altitude, it has been shown that, in the steady condition, the motion and pressure must be the same for equal latitudes and altitudes. If, therefore, meridional components of motion exist, the individual particles of air must move on closed surfaces of revolution about the earth's axis. Such a surface will contain, at any moment, all those particles that at any previous time crossed over any particular parallel circle of this surface; and the meridional component of motion must have the same direction along any meridional section of the surface. The space enclosed by one of these surfaces may enclose a part of the axis of rotation or not. The *first* case cannot occur, because on account of the general agreement in direction of the meridional component of motion, there would arise a steadily increasing accumulation of atmosphere at one pole of the surface of revolution, while from the opposite pole the adjacent air would steadily depart without any corresponding replacement by other air.

In the *second* case when the motion of the air occurs on a closed surface that does not intersect the earth's axis, we may refer back to the equation of continuity for fluid motion. This equation, referred to rectangular coördinates and for the case of steady motion, reads

$$\frac{\partial \mu \frac{dx}{dt}}{\partial x}+\frac{\partial \mu \frac{dy}{dt}}{\partial y}+\frac{\partial \mu \frac{dz}{dt}}{\partial z}=0.$$

Transform this equation to another system of coördinates r, φ, z, connected with x, y, z by the equations

$$x=r\cos\varphi,\quad y=r\sin\varphi,\quad z=z;$$

take the origin of coördinates at the centre of the earth, the axis of z as the earth's axis of rotation and recall that under the conditions here assumed, the derivatives with respect to φ are equal to zero, and we obtain:

$$\frac{\partial \mu \frac{dr}{dt}}{\partial r}+\frac{\partial \mu \frac{dz}{dt}}{\partial z}+\frac{\mu}{r}\cdot\frac{dr}{dt}=0;$$

whence it follows that

$$\mu r\frac{dr}{dt}=-\frac{\partial F(r,z)}{\partial z},\qquad \mu r\frac{dz}{dt}=+\frac{\partial F(r,z)}{\partial r}$$

and

$$\frac{\partial F(r,z)}{\partial z}\cdot\frac{dz}{dt}+\frac{\partial F(r,z)}{\partial r}\,\frac{dr}{dt}=0.$$

But a function $F(r,z)$ varying continuously in such a way that the equation $F(r,z)=C$ shall represent closed surfaces lying only within or without a semi-ellipsoidal shell, does not exist. Therefore in no case, when the distribution of temperature causes meridional components of motion can a condition of steady motion and stationary pressure exist within the atmosphere.

Excepting the memoirs of von Helmholtz, already referred to, the assumption of a stationary condition has hitherto been adopted in all investigations into the motions of the atmosphere, *e. g.*, Ferrel, "Meteorological Researches," Part I., 1877, and Oberbeck (*Berlin. Sitzungb.* 1888).* Certain errors in the two memoirs by myself (*Met. Zeits.*, 1893, pp. 1, 131) also result from this erroneous assumption. Such an assumption has certainly led to impossible conclusions, but the fundamental nature of the error has, so far as I know, not been recognized hitherto.

Understanding, therefore, that in an atmosphere under these ideal assumptions, a steady condition cannot occur, it follows that the movements of the real atmosphere under conditions that imply continuous values and such as depend only on the latitude and the altitude above the earth's surface, must consist of oscillations and *regularly* progressing waves. These waves have no relation to the billows considered by von Helmholtz, for the existence of these latter depends on the formation of surfaces of discontinuity whereas the present discussion is based wholly upon continuous conditions of motion and pressure.

The waves originating in the difference of temperature between the equator and the poles may be classified into two groups. The first group includes waves that advance in the direction of the meridian, and that may partly resolve into standing oscillations with nodes at the poles; the second group consists of waves that advance in the direction of the circles of latitude. The atmospheric pressures and motions resulting from the combination of these two groups of intersecting waves, give rise to the phenomena hitherto called cyclonic and anticyclonic.

If the heights of the waves that advance in the direction of the circles of latitude vary with the latitude, then this would cause so-called cyclonic and anticyclonic phenomena in the atmosphere, even though no waves moving along the meridians were formed. In any case, however, the preceding considerations lead to the following conclusion: A large part of the atmospheric phenomena called cyclones and anticyclones, having a progressive motion along the earth's surface, belongs to the general atmospheric circulation that develops in consequence of the differences of temperature between the equator and the poles; that is to say such phenomena would be found in the terrestrial atmosphere even if there were no differences of temperature along the individual latitude circles and no friction or resistances of any kind. These phenomena are therefore not to be considered as local disturbances of an equilibrium of the atmosphere produced by the temporary distribution of temperature between the equator and the poles, but present themselves as the regular, normal and mechanically necessary phenomena of an atmosphere for which, in general, no such state of equilibrium can exist except in very definite and restricted cases.

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* Translated in the "Mechanics of the Atmosphere," pp. 176, 178.

In several important memoirs (*Vienna Sitzungsb.*, *math. nat. Klasse*, vol. 99, Abt. IIa, p. 204; Vol. 101, Abt. 2a, p. 597; Vol. 102, Abt. 2a, p. 11 and p. 1369.) M. Margules has treated of atmospheric waves, by submitting to analytical investigation the *possible motions* of the air in a rotating spheroidal shell under certain assumptions made in order to simplify the calculations. But the results of Margules's investigations ought not to be applied without further consideration to an atmosphere of great height. According to Margules in the case of an advancing wave in a frictionless medium the strongest wind should occur simultaneously with the extremes of pressure.

Under the assumption, as before, of no friction and a distribution of temperature dependent only upon latitude and altitude there will be formed, besides the waves, a stationary zonal distribution of pressure corresponding to atmospheric currents parallel to the circles of latitude. This zone partially balances the disturbance of equilibrium due to a part of the temperature distribution, and must respond to the conditions under which equilibrium can occur in the atmosphere. If the meridional section shows that there are maxima and minima in this distribution of pressure, as appears to be actually the case, then these, in connection with the waves running along the zone of latitude, also lead to the formation of anticyclonic and cyclonic phenomena.

There have been many attempts to explain or compute this stationary distribution of pressure, as it is presented to us, mixed up with another pressure distribution due to thermal anomalies, in the mean values of atmospheric pressure, and that, too, in the belief that in these mean pressures, we have the simple result of the difference of temperature between the equator and the poles. The attempts to deduce this evident zonal distribution of pressure, even in its general features, from general dynamic considerations must even *a priori* be considered as failures. It is as impossible for general considerations to lead to this result as for them to determine the form of an uniformly rotating fluid, whose particles are subject only to the general attraction of masses, as that of an ellipsoid.

The mathematical figure of the rotating fluid and of the surfaces of equal pressure in our atmosphere (therefore also the location and value of the zonal maxima and minima of pressure on the earth's surface, where it intersects the surfaces of equal pressure), can be found if not by observation, then only by expressing numerically the relations to each other of the quantities that come into consideration.

Such numerical computations of zonal distributions of pressure have been executed especially by Ferrel and Oberbeck. Both these authors, as already remarked, assume a stationary condition (as to temperature, motion and pressure). This would, perhaps, not prevent their computation of the stationary zonal part of the distribution of pressure from being, at least, approximately satisfactory. But Ferrel applies the theorem of the conservation of areas to an individual particle of air, which is not allowable; Oberbeck assumes the surface of the earth to be spherical, and neglects the differences of density in the atmosphere—which are precisely those that bring about the motions of the air.* Hence the problem to determine numerically the stationary zonal distribution of pressure is not yet resolved.

The wave motion, existing even in an ideal atmosphere, will be composed of groups of waves of different lengths and altitudes, as well as of different periods. In reference to the waves moving along the latitude circles, it is easy to see that the length of these must be a simple fraction of the whole circumference of the circle. In this case the pressure and motion correspond to the sum of a stationary zonal distribution, and of that due to the individual waves in their temporary positions. The progressive movement of the individual maxima and minima is therefore dependent upon the progress of the individual waves; the variations in the depths of the minima and heights of the maxima, as well as the distribution of pressure in their neighborhood, depend on the differences in phase that the waves have with respect to each other at the places where the minima or maxima occur. Therefore, for example, the minimum deepens when the deepest parts of the different waves approach each other. It is easy to see that the isobars that surround the minima and maxima do not in general have a circular form. This form can only occur when the differences of pressure in the direction from west to east are the same as those that simultaneously occur in the direction from north to south, a case that can only seldom occur, as is confirmed by experience, and which, in consequence of the further progress of the waves, is soon converted into other relations. Hence it is seen that when circular cyclonic and anticyclonic phenomena are made the foundation of our views as to atmospheric motions and other forms of isobars are considered as departures from these normal conditions—this is not accordant with the actual process of nature.

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*Oberbeck neglects the changes of temperature that depend upon the time in order to confine himself to steady motions and stationary conditions; he also considers the atmosphere as an incompressible liquid, thereby neglecting vertical differences of density due to pressure, but he does not neglect the horizontal differences of density due to temperature. C. A.

In the atmosphere surrounding the earth, on account of the irregular distribution of water and land, the zonal distribution of atmospheric temperature hitherto considered does not exist, and, consequently, the actual distribution of the pressure and the movement of the air will differ from that corresponding to such a zonal distribution of temperature. It will, however, be proper to analyze the relations actually existing into two components, one corresponding to the mean temporary zonal distribution of temperature and the other depending on any departure from this mean temperature over a greater or smaller region.

The phenomena resulting from this latter departure, even when they extend over large regions, still have a local character and are dependent on the nature of the respective terrestrial surfaces beneath them, and will, therefore, not take part in those motions that are caused by the average zonal temperature conditions but will be added to or superposed upon them. The analysis of the phenomena into these two subdivisions appears so much the more justifiable since the local temperature irregularities diminish with altitude, and the zonal temperature distribution becomes by so much the more important. Under circumstances that are easy to perceive, this summation or superposition of the progressive waves due to the general circulation, and of the pressure and motion due to local irregularities of temperature, can also produce progressive cyclonic and anticyclonic phenomena.

It must not be forgotten that the movements depending on the local irregularities of temperature cannot remain perfectly stationary but must also contain undulatory phenomena, for the pressure and motion cannot in general assume such a status that the motion of the particles of air follows the intersections of the surfaces of equal partial pressure with the level surfaces due to the force of terrestrial gravitation.

But this latter constitutes the general condition by reason of which the motion of the air in the rotating atmosphere may remain the same, that is to say steady, in any finite region and during any finite time. Now this condition is, in general, not fulfilled in the atmosphere (neither by a purely zonal distribution of temperature, nor by the partial motions that result from local departures of temperature from the zonal averages), and, hence winds that are steady as to direction and force cannot persist during any considerable interval of time, but atmospheric billows or even waves of small length must be formed. The preceding deductions lead us to the variations of pressure and the gusts within storms that have lately been demonstrated by observations, even for great altitudes, especially by Vallot on Mont Blanc and by Pernter on Sonnblick.

These latter and the atmospheric waves that become visible in the cloud formations known as "mackerel" sky, are probably the shortest waves that we have occasion to observe. The atmospheric waves of next greater extent are those that play an important role in the formation of thunderstorms, and a remark by Erk (*Met. Zeit.* 1894, p. 271) in reference to the thunderstorm of June 7, 1894, offers an excellent illustration of these. Of about the same order of dimensions are the tornadic storms whose great intensity depends upon the conjunction of several wave troughs. From the point of view here presented, we can also understand how it happens that such phenomena disappear and again reappear at other points in the prolongation of the direction of their earlier paths. This phenomenon can be regarded as a consequence of the interference of systems of waves progressing with different velocities.*

We must, therefore, think of the atmosphere as filled with numerous systems of waves of extremely different sizes and especially, first, the regular waves that correspond to a certain average zonal distribution of temperature and, second, the irregular waves that result from the local deviations from this distribution of temperature and which extend upward to only a slight altitude above the earth's surface.

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*I must consider this explanation as wholly inapplicable to the tornadoes of the United States. C. A.

As another class of progressive changes of atmospheric pressure, there must be mentioned certain sudden changes of pressure whose initial development must be ascribed to the formation of discontinuities in the atmosphere. Although, as before stated, the formation of surfaces of discontinuity in the atmosphere, as imagined by von Helmholtz as an application to a ring of air of the theorem of conservation of areas, does not correspond to the principles of mechanics, still other possibilities can be imagined that would lead to such discontinuities, as, for example, the sudden precipitation of the aqueous vapor contained in any portion of the atmosphere. But such discontinuities will always extend over small regions only. The sudden change of pressure due to these can, indeed, continue and advance like a wave produced by concussion, even when the original cause of the development of the discontinuity itself has disappeared. The phenomenon described in the *Annalen d. Hydrographie*, 1889, p. 242, and which was traced from Sylt to Pola may belong to this class.

Since we, at present, do not possess a reliable mathematical or analytical presentation of the atmospheric pressures and motions, even under the simplest assumptions, therefore, the problem that first occurs is to analyze the atmospheric relations as derived from observations into those more regular features that belong to the general atmospheric circulation and those due to local departures and to investigate each of these by itself.

The more accurate knowledge of the regular phenomena of the general atmospheric circulation will reveal to us the connection of phenomena that are, geographically speaking, widely separated from each other, and will lead us to certain periodicities in the weather. It will thus be seen whether, besides the regular seasonal variations in the influence of the solar heat on the earth, there be still other causes lying outside of our planet that affect the terrestrial atmosphere.

Thus the question will then be answered, to what extent the solar spot period, by a change in radiant heat or in electrical phenomena, affects the processes in our atmosphere. For it may here be stated that within any rotating fluid envelope, whose parts are attracted, according to the general law of the attraction of masses, by a solid of revolution within the fluid, there cannot exist any stationary condition, or steady motion, when any influences, distributed zonally, prevent a rotation with uniform average, angular velocity of the whole envelope, or at least of those parts that are at equal distances from the axis of rotation. If, therefore, such influences exist on the sun, then wave formations must occur on it, and the sun spots, which have already been considered as vortex phenomena, as well as their variability, would thus find an explanation similar to that of the cyclones and anticyclones of the terrestrial atmosphere. It thus becomes quite possible that the regular phenomena of the general atmospheric circulation of the earth may, for purely mechanical reasons, have the same period as those on the sun. In this case there would not need to exist any direct connection between the sun spots and our weather, even though the same period should be established beyond doubt by observation. On the other hand, a further development of the theory may, perhaps, lead to conclusions as to certain mechanical relations, especially as to the velocity of rotation of the sun.

Even the discussion as to the influence of the moon on our weather, which is still by no means settled, will, in this way, be brought to an end, for it must then be possible to satisfactorily show whether the otherwise regular processes of atmospheric circulation are influenced by the changes in the moon's position, especially by its motion in declination.

It is to be hoped that the method here indicated may lead meteorology out of the region of vacillating ideas that now control it into a broader field, and place it among the exact sciences, where everything is reduced to numerical computation, and thus, to an important extent, further its application to daily practice.

DEUTSCHE SEEWARTE, HAMBURG.

SOME POINTS IN THE ELEMENTS OF THE THEORY OF FUNCTIONS.

BY PROFESSOR W. F. OSGOOD.

I. *A New Definition of an Analytic Function.* Cauchy defined $f(z)$ to be an analytic function of z when $f(z)$ is continuous and

$$\frac{f(z+\Delta z)-f(z)}{\Delta z}$$

converges toward one and the same limit when Δz converges in any manner toward 0. From this definition, coupled with a requirement concerning the derivative $f'(z)$, the fundamental theorem of his whole theory is obtained, namely: If $f(z)$ is analytic and $f'(z)$ continuous throughout a region, then $\int f(z)dz$ taken along any closed path lying in the region vanishes.

We may, however, equally well turn things about and take as definition the following:

DEFINITION: *$f(z)$ is an analytic function of z if $f(z)$ is continuous throughout a region and $\int f(z)dz$ taken along any closed path lying wholly within the region vanishes.*

For then the function

$$F(z)=\int_a^z f(z)dz$$

will be an analytic function according to Cauchy's definition, and hence by Cauchy's theory its derivative $f(z)$ will also be analytic.

Thus while Cauchy's definition is based on the infinite process of differentiation, the present definition is based on the infinite process of integration, the integral being defined as the limit of a sum. Let it here be emphasized that these two processes, in the conceptions and definitions on which they rest, are wholly coördinate with each other; for neither can it be claimed in this respect that it is more elementary than the other.

There is one essential point of difference between these definitions. Cauchy's definition is based on the behavior of the function *in a point;* this definition prescribes the behaviour of the function *throughout a region.* Weierstrass's definition of an analytic function is of the latter kind. Furthermore, Cauchy is obliged in order to get any theorems worth mentioning, to demand* that the *derivative* $f'(z)$ be continuous throughout a region. Here, that requirement is brought nearer home by being transferred to the definition and there applies to the *function* itself.

It seems reasonable to expect that from this definition some elementary theorems will be more directly deducible than from Cauchy's definition. As an example take the theorem that a uniformly convergent series of analytic functions defines an analytic function.†

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* One may, if one pleases, incorporate this demand into the definition; it seems to me undesirable to do so. But even then that which is essential in the present observation would not be changed.

† This application of the definition was called to my attention by Prof. Bôcher.

Let the series be

$$f_1(z)+f_2(z)+\cdots\cdots$$

Then its value $f(z)$ is a continuous function $f(z)$ and it remains to show that $\int f(z)dz$ taken along any closed path vanishes. This follows at once from the uniform convergence; the series can be integrated term by term and the integral of each term along the path in question vanishes.

It is to be noticed that this proof belongs to the most elementary class of proofs, in that it calls for no explicit representation of the functions entering (*e. g.*, by Cauchy's integral or by a power series).

II. A fundamental theorem in the elements of the theory of functions of a complex variable is the following:

(A) *If $f(z)$ is single-valued and analytic at all points*[*] *of a certain region except at the point $z=c$, and if $f(z)$ does not become infinite at c, then $f(z)$ is analytic in c also, or can be made so by the assignment to $f(z)$ in the point c of the value that $f(z)$ approaches when z approaches c.*

This theorem forms the basis for the theorems relating to the behavior of a single valued function $f(z)$ in the neighborhood of a singular point, P, $f(z)$ being analytic at all points in the neighborhood of P, namely, either (1) the function

$$\varphi(z)=(z-c)^m f(z), \qquad z \neq c; \qquad \varphi(c)=0$$

where m denotes a positive integer, is analytic in P; or (2) $f(z)$ comes arbitrarily near to any assigned value in every neighborhood of the point P.

The proofs of this theorem ordinarily given are either wrong or far less elementary than one could wish, in view of the nature of the theorem. It is the purpose of this paper to consider some of these proofs and to give a new one which is at the same time simple and elementary.

Riemann's Proof.—Riemann† first proves the following lemma: Let u be a harmonic function‡ at all points of a region except P; let

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[*] With the new definition of an analytic function given in I, by the expression "$f(z)$ is analytic in the point a" is to be understood that a lies within a region throughout which $f(z)$ is analytic.

† *Inaugural Dissertation*, Göttingen, 1851; each theorem is here stated in somewhat less general form than in the original.

‡ By *harmonic* is meant a function which together with its first and second derivatives is continuous and satisfies Laplace's equation: $\Delta u=\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}=0.$

$$\lim_{\substack{x=0\\y=0}} \rho \frac{\partial u}{\partial x}=0, \qquad \lim_{\substack{x=0\\y=0}} \rho \frac{\partial u}{\partial y}=0,$$

where ρ denotes the distance of the point (x, y) from P; and finally let u remain finite in the neighborhood of P; then u is harmonic in P also, or can be made so by assigning to u in the point P the value that u approaches when (x, y) approaches P in any manner. He then states the theorem (A), replacing the words, "if $f(z)$ does not become infinite at c" by "if $\lim\,(z-c)\,w=0$ when z approaches c", the function $f(z)$ being denoted by $w=u+vi$; and proves it as follows. He forms the integral

$$U=\int_{(x_0,\, y_0)}^{(x,\, y)} (u\, dx-v\, dy)$$

where v denotes the conjugate function of u, shows that its value is independent of its path and that the function U thus defined, which is continuous at P, is by the theorem just quoted harmonic at P also. And since

$$u=\frac{\partial U}{\partial x}, \qquad v=-\frac{\partial U}{\partial y}$$

the truth of the statement is evident.

The first of these theorems is interesting as showing how little stress Riemann laid on a systematic development of the theory of the logarithmic potential (*i. e.*, of harmonic functions) as a basis for the theory of functions of a complex variable. From this point of view the theorem (A) would be deduced from the theorem (B): If u is harmonic at all points of a certain region except at the point P and if u does not become infinite at P, then u is harmonic in P also, or can be made so by the assignment to u in the point P of its limiting value.

The Pseudo-Riemann Proof.—Riemann's rigorous proof of (A) has fared hardly with his expounders, whose version of it is as follows. Form the function

$$\varphi(z)=(z-c)\,f(z), \qquad z+c; \qquad \varphi(c)=0$$

Then $\varphi(z)$ is analytic in the neighborhood of c and continuous at c, *therefore* (sic) $\varphi(z)$ is analytic at c and can be developed by Taylor's theorem:

$$\varphi(z)=a_0+a_1\,(z-c)+a_2\,(z-c)^2+\cdots$$

But $\varphi(c) = 0$, hence $a_0 = 0$ and

$$\varphi(z) = (z-c)f(z) = a_1(z-c) + a_2(z-c)^2 + \cdots$$

or
$$f(z) = a_1 + a_2(z-c) + \cdots$$

This proof was given in the second edition of *Durège's Theorie der Functionen*, 1873, and has not been changed in the translation that has just appeared of the fourth edition. Curiously enough the proof given in the first edition, 1864, was rigorous, or at least could readily have been made so,— it is not complete. It was the proof given below as the first elementary proof and it seems likely that this proof was the one given by Riemann in his lectures. But in the improvements made in the second edition it was replaced by the one just cited. *Harnack*, whose desire frequently exceeded his ability to be rigorous, gave a similar proof,* likewise uncorrected in the English translation. *Forsyth* has also fallen into this error.† *Biermann*, in his attempt to expound Weierstrass's theory of functions, gives this same proof; ‡ this passage was sharply criticised in the *Fortschritte d. Math.*, vol. 19, 1887, p. 363.

Perhaps the reason for Durège's changing to the above proof is to be found in a paper by Roch, one of the earliest expounders of Riemann, which formed the second of a set of three papers on Riemann's theory, appearing in Schlömilch.†† In § 8 (vol. 8, p. 187) Roch expounds § 13 of Riemann's inaugural dissertation and there does one of two things: either he makes precisely the error above pointed out or he tacitly assumes Theorem (A). On some accounts the latter seems more likely to be the case; but this passage would easily account for Durège's proof.

Proof by Laurent's Theorem. This theorem says that if $f(z)$ is analytic at all points of a circular ring, *i. e.*, of a region included between two concentric circles with centres at c, then $f(z)$ can be expressed within this region in the form:

$$f(z) = \sum_{-\infty}^{\infty} a_\mu (z-c)^\mu$$

From this theorem the theorem (A) can be deduced. But this proof of (A), if given by Cauchy's methods, involves in addition to Cauchy's integral theorems the representation

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***Diff.- u. Integralrechnung*, § 189.

†*Theory of Functions*, p. 52.

‡*Analytische Functionen*, p. 174.

†† *Zeitschrift f. Math. u. Phys.* vol. 8, pp. 12–26 and pp. 183–203, 1863 and vol. 10, pp. 169–194, 1865.

of $f(z)$ by a power series and some properties of such series; if given by Weierstrass's methods, is still less elementary, the establishment of Laurent's theorem in that theory coming at a comparatively late stage. This subject is discussed at some length by Pringsheim in the last number of the *Math. Annalen*, who develops a number of elementary theorems, inclusive of the theorem (A), by the aid of power series plus the infinite process known as the *mean value*, a process which is essentially a restricted form of integration, and thus combining with the method of power series that which is essential in complex integration for the theorems in hand, he gives Cauchy's proofs of theorems in power series without the aid of Cauchy's integral.

Elementary Proofs of (A).—*First Proof.* Let C denote any curve enclosing the point c and lying within the region in question, and let t be any point but c within C. Surround c by a small circle γ not including t. Then by Cauchy's theorem

$$f(t)=\frac{1}{2\pi i}\int_C\frac{f(z)\,dz}{z-t}+\frac{1}{2\pi i}\int_\gamma\frac{f(z)\,dz}{z-t}$$

The second of these integrals vanishes, for $f(z)\,/\,z-t$ remains finite in the neighborhood of c, and when the radius of γ approaches 0, the length of the path of integration approaches 0. But the value of the integral is constant. hence it must be 0. Thus the formula

$$f(t)=\frac{1}{2\pi i}\int_C\frac{f(z)\,dz}{z-t}$$

represents the function $f(t)$ at all points within C except c. But the integral on the right hand side of this equation defines a function of t that is analytic in c too. Hence the theorem.

Second Proof. I now proceed to give a second elementary proof of (A), making use of the definition of I. Let

$$\varphi(z)=(z-c)f(z), \qquad z+c; \qquad \varphi(c)=0.$$

Then $\varphi(z)$ is analytic at c too; for $\varphi(z)$ is continuous throughout the region about c and $\int\varphi(z)\,dz$ taken along any closed path of this region vanishes, as will now be shown. Integration by parts gives

$$\int(z-c)\,f(z)\,dz=(z-c)\,F(z)-\int F(z)\,dz$$

where
$$F(z)=\int_a^z f(z)\,dz$$

is continuous and as is readily seen single-valued. Thus the first term on the right taken along any closed path of the region is 0. As to the second term, consider any closed path not including c. Then $F(z)$ and $F'(z)=f(z)$ are both continuous throughout the enclosed region and hence the integral along the boundary of the region is 0.* Finally let the path include c and surround c by a small circle. Then $\int F(z)\,dz$ extended in the usual way over the path and the circle will vanish. But the contribution that the circle yields is readily seen to be null; and thus $\int F(z)\,dz$ along any closed path whatever vanishes. Hence $\varphi(z)$ is analytic at c too. Its derivative at c is

$$\lim_{z=c} \frac{\varphi(z)}{z-c} = \lim_{z=c} f(z)$$

Hence $f(z)$ is continuous at c and thus fulfills both conditions of the definition of an analytic function given in I, and the proof is complete.†

HARVARD UNIVERSITY, *March* 1896.

ON THE MOTION OF A HOMOGENEOUS SPHERE OR SPHERICAL SHELL ON AN INCLINED PLANE, TAKING INTO ACCOUNT THE ROTATION OF THE EARTH.

BY PROFESSOR ALEXANDRE S. CHESSIN.

The influence of the rotation of the earth on the motion of bodies on its surface has been the subject of most interesting experiments, of which those with falling bodies and those of Foucault with the pendulum and the gyroscop may be mentioned. We propose to give here another interesting illustration of that influence, namely, on the motion of a homogeneous sphere or spherical shell on a plane inclined to the horizon. We will identify the system of axes (X, Y, Z) with the *absolute*, the system (Ξ, Υ, Z) with the *relative* space, and first establish the differential equations of relative motion of a spherical shell in general.

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* Cf. Goursat's proof of this theorem, *Acta Math.*, vol. IV.; Harkness and Morley, *Theory of Functions*, p. 164.

† Hölder has given in the *Math. Annalen*, vol. 20, 1882, a proof very similar to this. He integrates the given function $f(z)$ twice and shows that the second integral is analytic at c too. Hence he infers that $f(z)$ is analytic at c. I had obtained the proof here given, however, before Hölder's proof was known to me.

Let $\xi_\iota, \eta_\iota, \zeta_\iota$ be the relative coördinates of a point (i) of a moving body; M its mass; V_ξ, V_η, V_ζ the components on the axes Ξ, Υ, Z, of the principal vector of the *real* forces, *i. e.*,

$$(1) \qquad V_\xi = \sum_\iota F_{\iota\xi}; \quad V_\eta = \sum_\iota F_{\iota\eta}; \quad V_\zeta = \sum_\iota F_{\iota\zeta};$$

where $F_{\iota\xi}, F_{\iota\eta}, F_{\iota\zeta}$ are the components on the same axes of the *real* forces applied at a point (i) of the body; in like manner let $V_\xi', V_\eta', V_\zeta'$ and $V_\xi'', V_\eta'', V_\zeta''$ be the components of the principal vectors of the *fictitious* forces of moving space and of compound accelerations, *i. e.*,

$$(2) \qquad V_\xi' = \sum_\iota m_\iota j_{c,\xi,\iota}; \quad V_\eta' = \sum_\iota m_\iota j_{c,\eta,\iota}; \quad V_\zeta' = \sum_\iota m_\iota j_{c,\zeta,\iota};$$

$$(3) \qquad V_\xi'' = \sum_\iota m_\iota j_{s,\xi,\iota}; \cdot V_\eta'' = \sum_\iota m_\iota j_{s,\eta,\iota}, \quad V_\xi'' = \sum_\iota m_\iota j_{s,\zeta,\iota};$$

where m_ι denotes the mass of a point (i); $j_{c,\xi,\iota}, j_{c,\eta,\iota}, j_{c,\zeta,\iota}$, denote the projections on the axes Ξ, Υ, Z, of the acceleration of the moving space at the point (i) and $j_{s,\xi,\iota}, j_{s,\eta,\iota}, j_{s,\zeta,\iota}$, the projections on the same axes of the compound acceleration of that point; finally let R_ξ, R_η, R_ζ be the components of the principal vector of the reactions due to the conditions of constraint; then we can write down the following three differential equations of the motion of the centre of inertia (c) of the body:

$$(4)_1 \qquad M\xi_c'' = V_\xi - V_\xi' + V_\xi'' + R_\xi$$

$$(4)_2 \qquad M\eta_c'' = V_\eta - V_\eta' + V_\eta'' + R_\eta$$

$$(4)_3 \qquad M\zeta_c'' = V_\zeta - V_\zeta' + V_\zeta'' + R_\zeta$$

The motion of the body about its centre of inertia will be defined by three differential equations, containing the projections of the moment of momentum of the *absolute* motion on a system of moving axes, having their origin at the centre of inertia of the body but otherwise independent of the same. Let $\mathfrak{X}, \mathfrak{Y}, \mathfrak{Z}$ denote the new axes; $m_1^{(a)}, m_2^{(a)}, m_3^{(a)}$ be the components on these axes of the moment of *absolute* momentum about the centre of inertia of the body; M_1, M_2, M_3 be the components on the same axes of the moment of the *real* forces; $\Lambda_1, \Lambda_2, \Lambda_3$ the components of the moment of the forces of reaction; finally let $\omega_1, \omega_2, \omega_3$ be the components of the angular velocity (ω) of the moving system

($\mathfrak{X}$, $\mathfrak{Y}$, $\mathfrak{Z}$). Then the three differential equations which we were to establish are:

$$(5)_1 \qquad \frac{dm_1^{(a)}}{dt} = \omega_3 m_2^{(a)} - \omega_2 m_3^{(a)} + M_1 + \Lambda_1$$

$$(5)_2 \qquad \frac{dm_2^{(a)}}{dt} = \omega_1 m_3^{(a)} - \omega_3 m_1^{(a)} + M_2 + \Lambda_2$$

$$(5)_3 \qquad \frac{dm_3^{(a)}}{dt} = \omega_2 m_1^{(a)} - \omega_1 m_2^{(a)} + M_3 + \Lambda_3$$

Let now the body be a homogenous spherical shell; let H be its moment of inertia about a diameter; $p^{(a)}$, $q^{(a)}$, $r^{(a)}$ the components on the axes $\mathfrak{X}$, $\mathfrak{Y}$, $\mathfrak{Z}$ of its *absolute* angular velocity; then, as we know,

$$(6) \qquad m_1 = Hp^{(a)}\,;\ m_2 = Hq^{(a)}\,;\ m_3 = Hr^{(a)}.$$

Further, let $p^{(r)}$, $q^{(r)}$, $r^{(r)}$ be the components on the same axes of the *relative* angular velocity of the spherical shell; $m_1^{(r)}$, $m_2^{(r)}$, $m_3^{(r)}$ be the components of the moment of *relative* momentum about the center of inertia of the body; then

$$(7)_1 \qquad m_1^{(a)} = Hp^{(a)} = H(p^{(r)} + \omega_1) = m_1^{(r)} + H\omega_1$$

$$(7)_2 \qquad m_2^{(a)} = Hq^{(a)} = H(q^{(r)} + \omega_2) = m_2^{(r)} + H\omega_2$$

$$(7)_3 \qquad m_3^{(a)} = Hr^{(a)} = H(r^{(r)} + \omega_3) = m_3^{(r)} + H\omega_3$$

$$(8)_1 \qquad \frac{dm_1^{(a)}}{dt} = \frac{dm_1^{(r)}}{dt} + H\frac{d\omega_1}{dt}$$

$$(8)_2 \qquad \frac{dm_2^{(a)}}{dt} = \frac{dm_2^{(r)}}{dt} + H\frac{d\omega_2}{dt}$$

$$(8)_3 \qquad \frac{dm_3^{(a)}}{dt} = \frac{dm_3^{(r)}}{dt} + H\frac{d\omega_3}{dt}$$

Turning now to our problem, let O Z be the direction of the force of gravity; X O Y the plane of the horizon; $\Xi\,O\,\Upsilon$ the inclined plane forming an angle α with the horizon; $O\,\Xi$ be the line of slope; β the angle between the vertical plane Z O Ξ and the meridian of the point O; the positive axis $O\,Z$ be so taken as to make an acute angle with the direction of the force of gravity (*i. e.*, with O Z); finally let the axes $\mathfrak{X}$, $\mathfrak{Y}$, $\mathfrak{Z}$ be taken parallel to the axes Ξ, Υ, Z. Then we have

$$(9) \qquad \frac{d\omega_1}{dt} = \frac{d\omega_2}{dt} = \frac{d\omega_3}{dt} = 0$$

and therefore

$$(10)\qquad \frac{dm_1^{(a)}}{dt}=\frac{dm_1^{(r)}}{dt}\ ;\quad \frac{dm_2^{(a)}}{dt}\cdot=\frac{dm_2^{(r)}}{dt}\ ;\quad \frac{dm_3^{(a)}}{dt}=\frac{dm_3^{(r)}}{dt}$$

so that the equations (5) reduce to the following:

$$(11)_1\qquad \frac{dm_1^{(r)}}{dt}=\omega_3\, m_2^{(r)}-\omega_2\, m_3^{(r)}+M_1+\Lambda_1$$

$$(11)_2\qquad \frac{dm_2^{(r)}}{dt}=\omega_1\, m_3^{(r)}-\omega_3\, m_1^{(r)}+M_2+\Lambda_2$$

$$(11)_3\qquad \frac{dm_3^{(r)}}{dt}=\omega_2\, m_1^{(r)}-\omega_1\, m_2^{(r)}+M_3+\Lambda_3$$

The three equations (4) together with the three equations (11) determine the motion of the spherical shell relatively to the earth. The expressions for ω_1, ω_2, ω_3 are readily found to be:

$$(12)_1\qquad \omega_1=\omega\,(\sin\lambda\sin\alpha+\cos\lambda\cos\alpha\cos\beta)$$

$$(12)_2\qquad \omega_2=\omega\cos\lambda\sin\beta$$

$$(12)_3\qquad \omega_3=\omega(\sin\lambda\cos\alpha-\cos\lambda\sin\alpha\cos\beta)$$

ω being the angular velocity of the earth, and λ being the latitude of the point O.

In the following we will neglect terms of the order ω^2 of smallness and higher, because the assumption that the force of gravity is constant within the whole space in which the motion of the spherical shell takes place in our example, is justified only if the results be required with an approximation up to terms of the order of ω and not higher. We then find:

$$(13)\qquad V_\xi=Mg\sin\alpha\ ;\quad V_\eta=0\ ;\quad V_\zeta=Mg\cos\alpha$$

$$(14)\qquad V_\xi'=V_\eta'=V_\zeta'=0$$

$$(15)\quad V_\xi''=2M\omega_3\,\eta_c'\,;\ V_\eta''=-2M\omega_3\,\xi_c'\,;\ V_\zeta=-2M(\omega_1\,\eta_c'-\omega_2\xi_c')$$

Let R be the reaction of the plane on which the sphere is moving; F_1 and F_2 the components of the force of friction between the sphere and the plane; then

$$(16)\qquad R_\xi=F_1\ ;\quad R_\eta=F_2\ ;\quad R_\zeta=\mathrm{R}$$

and equations (4) become

$$(17)_1\qquad M\xi_c''=Mg\sin\alpha+2M\omega_3\,\eta'+F_1$$

$$(17)_2 \qquad M\eta_c'' = -2M\omega_3\,\xi_c' + F_2$$

$$(17)_3 \qquad 0 = Mg\cos a - 2M.(\omega_1\eta_c' - \omega_2\,\xi_e') + \mathrm{R}$$

If a denote the radius of the *exterior* surface of the spherical shell we shall have

$$(18) \qquad M_1 = M_2 = M_3 = 0$$

$$(19) \qquad \Lambda_1 = -aF_2\,; \quad \Lambda_2 = aF_1\,; \quad \Lambda_3 = 0$$

and equations (11) become

$$(20)_1 \qquad \frac{dp^{(r)}}{dt} - \omega_3\,q^{(r)} + \omega_2\,r^{(r)} = -\frac{a}{H}\,F_2$$

$$(20)_2 \qquad \frac{dq^{(r)}}{dt} - \omega_1\,r^{(r)} + \omega_3 p^{(r)} = \frac{a}{H}\,F_1$$

$$(20)_3 \qquad \frac{dr^{(r)}}{dt} - \omega_2 p^{(r)} + \omega_1\,q^{(r)} = 0$$

To make our problem a determinate one, it is necessary to indicate the way in which the spherical shell moves on the plane. We will assume that *it rolls on the plane without gliding*. Then the relative velocity of the point of contact of spherical shell and plane is zero, which gives the two equations of condition

$$(21) \qquad \xi_e' + a\,q^{(r)} = 0\,; \quad \eta_e' - a\,p^{(r)} = 0$$

If now we eliminate F_1 and F_2 from the seven equations $(17)_1$, $(17)_2$, $(20)_1$, $(20)_2$, $(20)_3$, (21), we obtain five equations between the five unknown quantities ξ_c, η_c, $p^{(r)}$, $q^{(r)}$, $r^{(r)}$, namely the three equations $(20)_3$ and (21) and the following two:

$$(22)_1 \quad (H + Ma^2)\frac{dg^{(r)}}{dt} = -Mag\sin a - (H + 2Ma^2)\omega_3\,p^{(r)} + H\omega_1 r^{(r)}$$

$$(22)_2 \qquad (H + Ma^2)\frac{dp^{(r)}}{dt} = (H + 2\,Ma^2)\,\omega_3\,q^{(r)} - H\omega_2\,r^{(r)}$$

A rigorous integration of these equations, although very simple, is not necessary, since we neglect terms of the order higher than ω. It is therefore preferable to integrate by successive differentiation, retaining every time only terms of the order not higher than ω, which gives

$$(23)_1 \qquad p^{(r)}=p_0^{(r)}+\frac{(H+2Ma^2)\,\omega_3 q_0^{(r)}-H\omega_2 r_0^{(r)}}{H+Ma^2}t-\frac{Mag\sin\alpha\,(H+2\,Ma^2)}{2\,(H+Ma^2)^2}\omega_3 t^2$$

$$(23)_2 \qquad q^{(r)}=q_0^{(r)}-\frac{Mag\sin\alpha}{H+Ma^2}t+\frac{H\omega_1 r_0^{(r)}-(H+2\,Ma^2)\,\omega_3 p_0^{(r)}}{H+Ma^2}t$$

$$(23)_3 \qquad r^{(r)}=r_0^{(r)}+(\omega_2 p_0^{(r)}-\omega_1 q_0^{(r)})\,t+\frac{Mag\sin\alpha}{2\,(H+Ma^2)}\omega_1 t^2$$

where $p_0^{(r)}$, $q_0^{(r)}$, $r_0^{(r)}$ are the initial values of $p^{(r)}$, $q^{(r)}$, $r^{(r)}$. If we assume that the initial relative angular velocity of the spherical shell is zero, the above formulas reduce to the following:

$$(24)_1 \qquad p^{(r)}=-\frac{Mag\sin\alpha\,(H+2\,Ma^2)}{2\,(H+Ma^2)^2}\omega_3 t^2$$

$$(24)_2 \qquad q^{(r)}=-\frac{Mag\sin\alpha}{H+Ma^2}t$$

$$(24)_3 \qquad r^{(r)}=\frac{Mag\sin\alpha}{2\,(H+Ma^2)}\omega_1 t^2$$

The values given by (23) or (24) being substituted in (21) a new integration gives immediately

$$(25)_1 \qquad \xi_c=\xi_{c0}-aq_0^{(r)}t+\frac{Ma^2g\sin\alpha}{2(H+Ma^2)}t^2-\frac{a[H\omega_1 r_0^{(r)}-(H+2\,Ma^2)\omega_3 p_0^{(r)}]}{2(H+Ma^2)}t^2$$

$$(25)_2 \qquad \eta_c=\eta_{c0}+ap_0^{(r)}t+\frac{a[(H+2\,Ma^2)\omega_3 q_0^{(r)}-H\omega_2 r_0^{(r)}]}{2\,(H+Ma^2)}t^2-\frac{Ma^2g\sin\alpha\,(H+2\,Ma^2)}{6(H+Ma^2)^2}\omega_3 t^3$$

We will give special attention to the case when $p_0^{(r)}=q_0^{(r)}=r_0^{(r)}=0$; if, besides, $\xi_{c0}=\eta_{c0}=0$ which can always be assumed by properly choosing the origin of the axes of coördinates, we obtain the following equations:

$$(26)_1 \qquad \xi_c=\frac{Ma^2g\sin\alpha}{2(H+Ma^2)}t^2$$

$$(26)_2 \qquad \eta_c=-\frac{Ma^2g\sin\alpha(H+2\,Ma^2)}{6\,(H+Ma^2)^2}\omega_3 t^3$$

i. e., the centre of the spherical shell describes a semicubic parabola.

In the case of a solid sphere $H=\frac{2}{5}Ma^2$ and

$$\xi_c=\frac{5}{14}g\sin\alpha\, t^2 \tag{27$_1$}$$

$$\eta_c=-\frac{10}{49}g\sin\alpha\,\omega_3\, t^2 \tag{27$_2$}$$

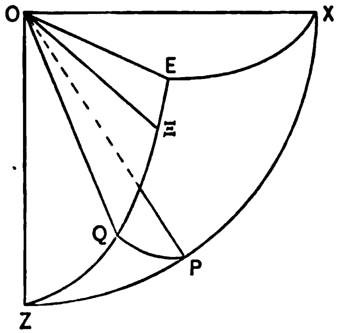
and therefore *the motion of the centre of a solid sphere does not depend on its mass or dimensions.*

We see that *the effect of the rotation of the earth on the motion of the centre of the sphere or spherical shell is to deviate it from the line of slope. This deviation may take place as well towards the east as towards the west of the meridian of the place* according as ω_3 is negative or positive, *i. e., according as* $\operatorname{tg}\lambda < \operatorname{tg}\alpha\cos\beta$ *or* $\operatorname{tg}\lambda > \operatorname{tg}\alpha\cos\beta$. *If* $\operatorname{tg}\lambda=\operatorname{tg}\alpha\cos\beta$ *the rotation of the earth has no influence on the motion of the centre of the sphere or spherical shell.*

These results may be interpreted geometrically as follows: Let OP be drawn parallel to the axis of the earth (toward the south) and let OQ be the projection of OP on the plane $\Xi\, O\, Z$. If then μ denote the angle of this projection with the horizon, we have

$$\operatorname{tg}\lambda=\operatorname{tg}\mu\cos\beta$$

Hence, if $\alpha<\mu$ *the deviation of the sphere or spherical shell from the line of slope takes place toward the west; if* $\alpha>\mu$ *this deviation takes place toward the east; and if* $\alpha=\mu$ *the sphere or spherical shell will move along the line of slope.*



$\sphericalangle XOE=\beta \quad \sphericalangle EO\Xi=\alpha \quad \sphericalangle EOQ=\mu \quad \sphericalangle XOP=\lambda$

We have tacitly assumed that $-\frac{\pi}{2} < \beta < \frac{\pi}{2}$. However, the investigation of the different cases which may present themselves, although interesting, would carry us out of the limits of this paper. Let only one remark be made with respect to the variation of the angle β, namely, that *the motion of the sphere or spherical shell depends on the orientation of the inclined plane and the deviation from the line of slope is maximum when* $\beta=\pi$, *minimum when* $\beta=0$.

NOTES.

A **Regular** meeting of the American Mathematical Society was held in New York on Saturday afternoon, April 25, at three o'clock, Professor **Mansfield Merriman** in the chair. There were twenty-one members present. On the recommendation of the Council, the following persons, previously nominated, were elected to membership: Mr. **F. M. McGaw**, Bordentown Military Institute, Bordentown, N. J.; Mr. **A. B. Chace**, Valley Falls, R. I.; Professor **E. D. Roe**, Oberlin College, Oberlin, Ohio. Four nominations for membership were received. The following papers were read:

(1) Dr. **Emory McClintock**: "On the most perfect forms of magic squares, with methods for their production."

(2) Professor **Charlotte Angas Scott**: "On the theory of the birational transformation of a plane curve."

The committee appointed by the Council to arrange for holding a colloquium in connection with the summer meeting at Buffalo, announces two courses of six lectures each, one by Professor Maxime Bôcher on the subject of Linear Differential Equations and their Applications, the other by Professor James Pierpont on the Galois Theory of Equations. The meetings of the Colloquium will begin immediately after the adjournment of the Society and will continue through the week. A special circular giving detailed information in regard to the arrangements for the Colloquium will be issued by the Committee early in July.

A **new** list of officers and members of the Society has been compiled by the Secretary and has been published by the Society. In the same pamphlet are included the Constitution and By-Laws of the Society, an account of the last Annual Meeting, and the reports of the Secretary and the Treasurer for 1895. Extra copies can be obtained from the Secretary.

PROFESSOR J. GRIESS, of Algiers, has translated into French Professor KLEIN's very interesting pamphlet, *Lectures on certain questions of elementary geometry*, reviewed by Miss SCOTT in the March number of the BULLETIN, pp. 157–164. The translator has expanded the text in a few places, and has substituted synthetic proofs for certain analytical proofs of the original. Nony & Company of Paris are the publishers of this translation.

COLUMBIA UNIVERSITY. The Department of Mathematics will give during the next academic year the following graduate courses, each course consisting of three lectures a week: by Professor T. S. Fiske, Special topics in the differential and integral calculus, first term; Differential equations and applications, second term; Theory of functions (second course), both terms. By Professor F. N. Cole, Theory of the complex variable, first term; Elliptic functions, second term. By Dr. J. B. Chittendon, Theory of invariants, second term. By Dr. J. E. Hill, Higher plane curves, first term; analytical theory of curves of double curvature and curved surfaces, second term. By Dr. G. H. Ling, Theory of numbers, both terms.

NEW PUBLICATIONS.

I. HIGHER MATHEMATICS.

AMERICAN MATHEMATICAL SOCIETY. See MATHEMATICAL PAPERS.

BOLZA (O.). See MATHEMATICAL PAPERS.

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CAUCHY (A.). Oeuvres complètes, publiées sous la direction scientifique de l'Académie des sciences et sous les auspices du Ministre de l'instruction publique, avec le concours de MM. Valson et Collet. 1re série, Vol. IX: Extraits des *Comptes rendus* de l'Académie des sciences. 1896. 4to. Fr. 25.00.

COLLET. See CAUCHY (A.).

DEL RE (A). Geometria projettiva ed analitica. Fascicoli 1 e 2. Modena, 1896. 8vo. pp. 1–256. Complete Fr. 8.50.

DUHAMEL (J. M. C.). Des méthodes dans les sciences de raisonnements. 3e édition. (En 5 parties). Partie II: Application des méthodes à la science des nombres et à la science de l'étendue. Paris, Gauthier-Villars, 1896. 8vo. Fr. 7.50.

EBNER (F.). Zur Theorie der Spiralflächen. [Diss.] Rostock, 1895. 8vo. 35 pp.

EULER (L.). Zwei Abhandlungen über sphärische Trigonometrie: Grundzüge der sphärischen Trigonometrie, und Allgemeine sphärische Trigonometrie, 1753–1779. Aus dem Französischen und Lateinischen übersetzt und herausgegeben von E. Hammer. Leipzig, 1896. 8vo. 65 pp. Cloth. Mk. 1.00

FORSYTH (A. R.). A treatise on the theory of functions of a complex variable. New York, Macmillan, 1896. 8vo. 22 and 682 pp. Cloth. $6.50

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GIUDICE (F.). See KLEIN (F.).

GRAHAM (J.). An elementary treatise on the calculus for engineering students, with numerous examples and problems worked out. London, Spon, 1896. 8vo. 266 pp.

HAMMER (E.). See EULER (L.).

HATHAWAY (A. S.). A primer of quaternions. New York, Macmillan, 1896. 12mo. 8 and 113 pp. Cloth. $0.90

HEIBERG (I. L.). See SERENUS ANTISSENSIS.

JORDAN (C.) Cours d' analyse de l'École polytechnique. 2e édition, entièrement refondue. Vol. III: Calcul intégral; équations différentielles. Paris, Gauthier-Villars, 1896. 8vo. 12 and 543 pp. Fr. 15.00

KLEIN (F.). L'œuvre géométrique de Sophus Lie. Traduit de l'anglais par L. Laugel. [*Nouvelles Annales de Mathématiques*, 3e série, Vol. 15, pp. 1–20].

————. Conferenze sopra alcune questioni di geometria elementare, redatte da F. Tägert, e tradotte dal tedesco da F. Giudice, con una prefazione di G. Loria. Torino, Rosenberg e Sellier, 1896. 8vo. 12 and 71 pp. Fr. 3.00

LATOON (F.). On common and perfect magic squares; with examples. London, 1896. 8vo. 140 pp. Cloth. 13s.

LAUGEL (L.). See KLEIN (F.).

LIE (S.) und SCHEFFERS (G.). Geometrie der Berührungstransformationen. Vol. I. Leipzig, Teubner, 1896. 8vo. 12 and 694 pp. Mk. 24.00

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MASCHKE (H.). See MATHEMATICAL PAPERS.

MATHEMATICAL PAPERS read at the International Mathematical Congress held in connection with the World's Columbian Exposition, Chicago, 1893. Edited by the Committee of the Congress: E. H. Moore, O. Bolza, H. Maschke, H. S. White. (*Papers published by the American Mathematical Society*, Vol. I.). New York, Macmillan, 1896. 8vo. 16 and 412 pp. Cloth. $4.00

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NEWSON (H. B.). Continuous groups of projective transformations treated synthetically. 8vo. [*Kansas University Quarterly*, Vol. 4. 1895. pp. 71-92, 243-249.]

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PEANO (G.). Notions de logique mathématique. [*Association française pour l'avancement des sciences*, Congrès de Caen, 1894.] 8vo. 5 pp.

——— Sulla definizione di integrale. [*Annali di matematica pura ed applicata*, Vol. 23.] 4to. 4 pp.

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STOLZ (O.). Grundzüge der Differential- und Integralrechnung. (In 2 Theilen.) Theil II: Complexe Veränderliche und Functionen. Leipzig, Teubner, 1896. 8vo. 10 and 338 pp. Mk. 8.00

TÄGERT (F.). See Klein (F.).

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TRAUB (K.). Der verjüngte Magister matheseos. Ein Beitrag zur Sphärik und absoluten Geometrie. Lahr, Schauenburg, 1896. 8vo. 4 and 12 pp., 1 plate. Mk. 0.50

VALSON (C. A.). See **CAUCHY** (A.).

VERONESE (G.). Intorno ad alcune osservazioni sui segmenti infiniti e infinitesimi attuali. (*Mathematische Annalen*, Vol. 47, pp. 323-432.)

WHITE (H. S). See **MATHEMATICAL PAPERS.**

II. ELEMENTARY MATHEMATICS.

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Byerly (W. E.). See **Chauvenet** (W.).

Chauvenet (W.). Plane Geometry, abridged and simplified by W. E. Byerly. New edition. Philadelphia, Lippincott, 1896. 8vo. Cloth. $0.40

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Exercices de géométrie en rapport avec les Éléments. Livre du maître, comprenant l'exposé des méthodes géométriques et 2000 questions résolues. 3e édition, revue et augmentée. Paris, 1896. 8vo. 20 and 1136 pp. Cloth. Fr. 16.00

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41 East 49th Street, New York City.

The following dates have been fixed for the meetings of the Society:

		P. M.			P. M.
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"	Nov. 28, "	"	"	Mar. 27, "	"
Wed.,	Dec. 30, "	"	"	April 24, "	"
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The meeting of Wednesday, December 30, 1896, will be the Annual Meeting of the Society for the election of Officers.

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OF THE

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A HISTORICAL AND CRITICAL REVIEW
OF MATHEMATICAL SCIENCE.

EDITED BY
THOMAS S. FISKE, ALEXANDER ZIWET, FRANK MORLEY.

2ND SERIES, VOL. II. NO. 10.

JULY, 1896.

NEW YORK:
PUBLISHED FOR THE SOCIETY
BY THE MACMILLAN COMPANY.
1896.

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PUBLISHED MONTHLY, EXCEPT AUGUST AND SEPTEMBER.

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THE ALGEBRAIC FUNCTIONS AND THEIR INTEGRALS.

Théorie des fonctions algébriques et de leurs intégrales. Étude des fonctions analytiques sur une surface de Riemann. By Paul Appell and Édouard Goursat. With a preface by Ch. Hermite. Paris, Gauthier-Villars et Fils, 1895. 8vo. 22+530 pp.

The general theories of functions that originated respectively with Cauchy, with Weierstrass and with Riemann have each found in the algebraic functions and their integrals an instructive example, the study of which has turned out to be so fruitful that the results of this research occupy a place of much importance in modern mathematics. Until recently the works of Neumann* and Clebsch and Gordan† have been the principal introductory treatises in this field. The object of the first of these was to set forth Riemann's theory, that of the second, as Gordan expresses it, "to build the bridge between the theory of the Abelian integrals and the theory of the higher plane curves." In the three decades that have elapsed since the appearance of these works, the activity of mathematicians in this field has been principally in research and not until recently has anyone been found to undertake the task of selecting and arranging the methods and the theorems that lie at the bottom of this subject, and working out *a systematic presentation of those things that the beginner ought to learn first.* With this purpose in view Messrs. Appell and Goursat applied themselves to the elaboration of their treatise and the result is a work that occupies a place of first importance among the many recent text-books that treat this subject to a greater or less extent.

In the preface Hermite enumerates the principal points of view that have been of importance in the development of the theory of the Abelian integrals, and defines the object of the present work.

The plan of the book is as follows. It is assumed that the reader is acquainted with the rudiments of the theory of functions and a syllabus of the theorems premised forms a nine page introduction. The hyperelliptic equation

---

* *Vorlesungen über Riemann's Theorie der Abelschen Integrale*, Teubner, first ed., 1865. The physical ideas underlying Riemann's theory were treated by Klein: *Ueber Riemann's Theorie der algebraischen Functionen und ihrer Integrale*, Teubner, 1882.

† *Theorie der Abelschen Integrale*, Teubner, 1866.

$$u^2 = A\,(z-e_1)\,(z-e_2)\cdots\cdots(z-e_n)$$

with its two-leaved Riemann surface presents an example of an algebraic equation of much didactic value; for since the functions that occur in the general case admit here explicit expression in simple form, the algebraic difficulties of the general theory are diminished, and still a degree of generality remains sufficient for purposes of illustration. This hyperelliptic example is discussed at great length and with perfect clearness, so that even the meagre knowledge of the theory of functions demanded of the beginner will suffice. But there is no abrupt transition in the character of the treatment when the general theory of the algebraic functions and their integrals is entered on. Jacobi's problem of inversion, normal curves, the moduli of an algebraic configuration,—these are among the last subjects treated, and thus the line is drawn below the theta functions.

We cannot feel that the omission of all mention of conformal representation is fortunate and we should have been glad to see the study of rational functions and their integrals on an algebraic configuration given a more geometric form by the earlier introduction of higher plane curves. And this for the reason that both of these geometric methods are elementary in their nature and well adapted to give the beginner broader views and a deeper insight into the theories treated.

With regard however to subjects of a more special nature we must acknowledge our indebtedness to Messrs. Appell and Goursat no less for what they have left out of their treatise than for what they have put in. For example, we were gratified to find no homogeneous variables under the sign of integration in so elementary a work. It is a temptation to which pioneer authors are subject to attempt to write at once a text-book and a compendium. Our authors have been content to write a text-book. We congratulate them on the success with which they have accomplished their task; we congratulate the younger students of mathematics on the acquisition of a valuable introduction to an important field of knowledge.

CHAPTER I. The Riemann surface is introduced in the usual way by the example $u^2=z$. The construction of the surface is clearly and carefully described and the surface is projected on the sphere. The elliptic and the hyperelliptic surfaces follow, the latter being defined by the equation

$$(1) \qquad u^2 = A(z-e_1)(z-e_2)\ldots\ldots(z-e_n),$$

and are treated in a similar manner. These first 24 pages give with remarkable clearness the conception of the Riemann surface. The surfaces considered from now on till Chapter IV are the surfaces (1). The residue in an ordinary point or a branch point (finite or infinite) is defined by means of the coefficient of a certain term in the expansion of the function into a power series, the relation between the definitions in the four kinds of points that thus arise being by no means obvious; and it is subsequently shown that the residue is always equal to $\int f(z)dz$ taken along any closed path about the point in question. In a treatment wedded to the consistent use of power series such a definition is explicable; but one is at a loss here to see why the residue is not defined by the integral and then evaluated in terms of the above coefficients. The behavior of a rational function $R(u,z)$ on the Riemann surface is studied and it is shown that conversely every single-valued function having no essentially singular points is a rational function of u, z. An elegant application of the method of residues yields a proof of the theorem that such a function has as many zeros as poles and hence takes on every value the same number of times. § 25 is incorrect, as is shown by the example

$$u^2 = z^4 - 1,$$

$$v_1 = z^2 + u_1, \qquad v_2 = z^2 + u_2, \qquad v_1 v_2 = 1$$

Weierstrass's definition of the *deficiency* of an algebraic configuration is given. Rational functions on the hyperelliptic configuration (1) with arbitrarily assigned poles are built up and it is shown by the aid of the explicit expressions obtained that the number of poles must in general be at least equal to $p+1$, where $n = 2p+1$ or $2p+2$.

CHAPTER II is entitled: Hyperelliptic Integrals, and deals with the integrals of rational functions of u, z on the Riemann surface belonging to (1). Such integrals are functions having at most poles and logarithmically singular points. The most general integral of the first class (everywhere finite, *i. e.*, having no singular points) is expressed in terms of the p linearly independent integrals

$$\int_{(z_0, u_0)}^{(z, u)} \frac{z^k\,dz}{u}, \quad k=0, 1, \cdots p-1.$$

The integral of the third class $\bar{\omega}_{a,b}^{a',b'}$ with logarithmic discontinuities at (a, b), (a', b') is formed for all possible positions of the points (a, b), (a', b') and for both an odd and an even n, and in § 38 a form of such an integral is given (containing however a foreign parameter λ) that embraces all of these special cases. A reference at this point to Klein's * normal form of the integral of the second and the third class would not have been out of place, though there are obvious reasons for not discussing these forms in the text. Now come the integrals $\zeta^{(\nu)}$ of the second class with a single pole of order $\nu+1$ and the discussion of the relations existing between these integrals and ω. The treatment here given is careful and exhaustive, but the beginner will need the guidance of a teacher, who will present to him the general case (where the point (a, b) is neither a branch-point nor infinity) and then give him the special cases to work out for himself. The chapter closes with the theorems regarding the expression of rational functions of u, z and their integrals in terms of the integrals of the three kinds already considered.

Chapter III is entitled: Connection of Two-Leaved Surfaces. The simplification in the analytical work of these first three chapters due to the explicit expressions always obtained for the functions studied finds its geometric counterpart in the avoidance of all general investigations in the field of analysis situs having for their object the proof of the existence of a set of cuts that render the Riemann surface simply connected, such a set of cuts being drawn here by inspection. But while the analytic treatment is always complete, there remain here certain inverse theorems that await their solution in Chapter V; for example, the theorem that p is the same number for all possible ways of dissecting the surface. The objects that analysis situs in its application to Riemann surfaces has in view are clearly set forth in these pages, which form an excellent introduction to the subject. By the aid of Cauchy's integral theorem (§ 59) extended to regions lying in more than one sheet of the Riemann surface,—in particular, the region T' obtained by drawing such a set of cuts in the surface T belonging to (1) as to render it singly connected,—single-valued functions on such regions are defined and thus the integrals of rational functions of u, z on T are studied with regard to their moduli of periodicity, the bilinear relations existing between these latter being established. Riemann's normal integrals of the first kind, together with the corresponding normal integrals of the second and third kinds are introduced and the periods of the latter are evaluated.

With Chapter IV, p. 165, begins the general treatment of

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* *Hyperelliptische Sigmafunctionen*, *Math. Ann.* vol. 27, 1886.

algebraic functions, *i. e.*, of the function u defined by the equation $F(u, z)=0$, where F denotes any polynomial in u, z. The chapter opens with Cauchy's proof as given by Briot and Bouquet that such a function is in general continuous; and it is further shown that it has in general a continuous derivative. If z is confined to a region including no singular point, the m values u can be grouped into m single-valued continuous functions or *branches*. Then comes the establishment of the form of development into power series about any point z_0, ordinary or singular, finite or infinite, together with the converse theorem (p. 177). Now it seems to us that the presentation would have gained materially at this point by the use of conformal representation and the Riemann surface. Thus the Riemann surface would be constructed in the neighborhood of the branch-point $z=a$ and transformed on the single-leaved plane by the relation $z-a=z'^p$, the function u becoming thereby a single-valued function of z'.* Useful symmetry in the form of representation of the algebraic configuration in the neighborhood of the point $z=a$, $u=b$ results from the introduction of a parameter t. We then have

$$z-a = t^p P_1(t), \qquad u-b = t^q P_2(t)$$

where p, q are positive or negative integers and $P_1(0)$, $P_2(0)$ do not vanish. Geometrically this form means the emphasizing of the fact that the u-surface in the neighborhood of $u=b$, as well as the z-surface in the neighborhood of $z=a$, is transformed conformally on the t-plane in the neighborhood of $t=0$. This form is so useful in the study of the behavior of a rational function of u, z (or of its integral) in a point of the surface, for example in the definition of the residue, the number of zeros or poles, the determination of the nature of the singular points of the integral $\int R(u, z)dz$, etc., that it seems unfortunate that the authors did not see fit to introduce it. In § 85 it is shown that if $F(u, z)$ is an irreducible function of u, z and $u_1, u_2, \cdots u_m$ are the values of u corresponding to an ordinary point z_0, then there must be a path for z starting at z_0 and ending at z_0, that will carry u_1 over into any other u, u_i The proof is based most simply on the theorem that if $F(u, z)$ and $\varphi(u, z)$ are two rational integral functions of the two independent variables u, z and if F vanishes whenever φ vanishes, then F and φ have a common factor. This is indeed a theorem of great importance and frequent application. But can it be assumed that students for whom it is necessary that the subject of this work be treated in such detail, are so familiar with it that it need not even be quoted? And if this theorem be not assumed, then it is expecting a great deal of the student to require him to supply the proof by the theory of functions. § 86 treats the method of loops. Now follows Puiseaux's method of treating the multiple branch-points of the algebraic configuration. In §§ 92, 93 the Riemann surface for any irreducible algebraic configuration is defined. The treatment begins with well chosen examples and is admirably clear, in fact, one of the clearest that exists. The residue of a function in any point of the surface is defined and many of the theorems of Chapter I are extended to the case of any irreducible surface,—for example, the theorem that a single-valued function on a Riemann surface, having no essentially singular points is a rational function of u, z. The chapter closes with the proof that branch-points of higher order may be regarded as the limiting case in which several simple branch-points have become coincident.

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*Cf. Neumann, Chaps. IV, V.

Chapter V begins with a general treatment of analysis situs, principally in two dimensions, modelled after Neumann. The presentation is in general clear, but the intuitional hypotheses might well have been more sharply marked. The theorem already referred to regarding p is established, Riemann's formula connecting the number of leaves and branch-points with the deficiency: $w=2(m+p-1)$ is deduced, and Riemann's canonical system of cuts is given. Examples follow. The deficiency of the binomial equation $u^m=R(z)$, where R denotes a rational function, is determined and all such equations for which $p=0$ or 1 are reduced to normal forms. In the course of this work a more general question is suggested—that of determining all "regular" Riemann surfaces of a given deficiency and at this point the surfaces belonging to the regular bodies ($p=0$) appear. The behavior of an Abelian integral, *i. e.*, of the integral of a rational function on the Riemann surface, in the neighborhood of any point is studied, such integrals are shown to have moduli of periodicity as in the hyperelliptic case, the integrals of the three classes are defined, and it is shown that there cannot exist more than p linearly independent integrals of the first class.

Chapter VI is devoted to rational transformations. Let $f(z, u)=0$ be an irreducible algebraic equation; let

$$Z=\varphi(z, u), \qquad U=\psi(z, u)$$

be two rational functions of z, u having respectively μ and ν

poles on the surface belonging to $f=0$. If z, u be eliminated between these three equations, the result is an algebraic equation $F(Z, U)=0$ which expresses the necessary and sufficient condition that the above three equations admit a common solution. Thus the chapter begins. Few subjects in elementary mathematics are so deceptive as that of elimination. The student easily mistakes familiarity with the formal operations of combining equations so as to get rid of certain quantities for a knowledge of the theory of elimination and utterly fails to grasp what the resulting equation at which he has arrived really signifies. Moreover, in the present case the quantities that enter are capable of becoming infinite (since homogeneous variables are not used) and thus even the algebraic treatment fails at this point. It therefore seems to us unfortunate that the processes employed should not have been illustrated geometrically by the interpretation of the above equations as plane curves, Z, U being parameters, and the way thereby pointed out to the analytical treatment by the aid of the theory of functions.—The theorem that the deficiency of a Riemann surface is invariant of a rational transformation is proved both geometrically and analytically and the effect of such transformation on the Abelian integrals is considered. A transformation of Halphen's is given by which any irreducible algebraic curve can be transformed into a curve having only linear cycles and the same transformation is used to prove Noether's theorem that any irreducible curve can be transformed rationally into a curve having only multiple points with distinct tangents, in particular only ordinary double points. The chapter closes with the algebraic representation of the coördinates of curves of deficiency 0, 1 and 2 by means of a parameter, and the transformation of curves of deficiency 0 and 1 into themselves.

Chapter VII begins with the subject of Normal Integrals. The form of the integrand of any everywhere finite integral is determined for any irreducible curve $F(z, u)=0$, the only restriction being that certain coördinate axes be avoided:

$$\int \frac{Q(z,u)}{F'_u(z,u)}\,dz$$

where Q is a polynomial of degree $m-3$ in z, u; and it is shown that there always exist p such integrals, linearly independent of each other. Picard's proof of this fundamental theorem is also given. An adjoint curve of degree $m-3$ cuts the ground curve in general in a group of fixed points and a group of variable points. In § 142 it is shown that this latter group is invariant of one-to-one transformation of the ground curve. Near the end of the section it is assumed that if a curve has only ordinary double-points, the adjoint curve will cut it in $2p-2$ points distinct from the double-points. This theorem is proved later in § 174; but at present it is not clear that the adjoint curve will not of itself go through some of the double-points oftener than it has been required to do. The integrals of the second and third classes are formed, the algebraic configuration being represented geometrically as a plane curve, while the integrand is built up by the aid of other curves intersecting in certain systems of points. The same geometrical method might well have been introduced into Chapters II and III, either as here, as a means of getting the integrals or merely for the purpose of illustrating the results there obtained.* Riemann's normal integrals of the first class are introduced and many theorems of Chapters II and III are generalized. A reference for the "well known theorem" of § 154 would not have been out of place.—Any Abelian integral I can be expressed in the form

$$I = \Phi\,(z, u) + I_1 + I_2,$$

where Φ denotes a rational function of z, u, I_1 an integral having only logarithmic singularities and I_2 an integral having at most poles of the first order in p points that can be chosen independently of the particular integral I. The functions Φ, I_2 are discussed to some extent for the case that logarithmic singularities are absent: $I_1 \equiv 0$ and the conditions that the integrand of I must satisfy that the integral I should reduce to a rational function of z, u are obtained. An interesting example is given. Next the question is proposed: suppose $\Phi \equiv 0$, $I_2 \equiv 0$; when can I_1 be expressed as a sum of logarithms of rational functions and an integral of the first class? and discussed at considerable length. The reduction of the Abelian integrals to integrals on surfaces of lower deficiency is illustrated by examples.

In CHAPTER VIII single-valued functions on a Riemann surface are treated by the aid of the Abelian integrals on the surface. The law of Riemann-Roch and Brill-Noether's law of reciprocity are established. The presentation of this important subject leaves nothing to be desired in regard either to clearness or elegance—at least, this is true if the reader can be trusted to form simple examples for himself, to render concrete the general theorems. *E. g.*, let the ground-curve be a C_4 or C_5 with 0, 1, or 2 double-points; where will the groups of points lie for which $\sigma > 0$? Applications follow. Hyperelliptic configurations are defined and referred to (1) of Chapter I as normal form: $u^2 = R(\lambda)$, where R denotes a polynomial. Two hyperelliptic configurations can be transformed rationally the one into the other when the *binary* forms

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* For a statement of the reasons for this suggestion see end of the review of Chapter IX.

$$\lambda_2^{2p+2} R\left(\frac{\lambda_1}{\lambda_2}\right), \quad \mu_2^{2p+2} R_1\left(\frac{\mu_1}{\mu_2}\right)$$

can be transformed linearly into each other. This requires the equality of the $2p-1$ absolute invariants of R, R_1,* and hence the hyperelliptic configuration has $2p-1$ *moduli.* The representation of rational functions by the aid of integrals of the third class is given.

CHAPTER IX is entitled : Abel's Theorem. The theorem is first proved by evaluating in two ways a certain integral extended over a closed path. This proof is brief and rigorous, and extremely simple in form; but it is open to the objection to which most proofs by Cauchy's integral are subject, that the thought underlying the theorem is hidden. For this reason the proof as originally given by Abel is to be preferred. This proof is given later, § 189. The proof of § 182 is incomplete; it remains to show that when curves of the sort f_1 are interpolated between f and φ, the corresponding ε's will never converge toward 0, and hence that one can pass from f to φ by means of a finite number of curves f_1. The addition theorem for the Abelian integrals is deduced and a set of Abelian differential equations is integrated. Finally, Abel's theorem is extended to space curves. We should have been grateful to the authors if they had seen fit to take up somewhere in this chapter, perhaps as introductory example, Abel's theorem for the integral of the first kind on the curve

$$u^2 = 4z^3 - g_2 z - g_3$$

or

$$u^2 = (1-z^2)(1-k^2 z^2),$$

the treatment being put into such form as to yield at once the addition theorem for this elliptic integral. It is true

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*A reference for this theorem ought certainly to have been given. See, for example, Clebsch: *Binäre Formen*, pp. 365 and 421. Also, for $p=1$, Klein: *Modulfunctionen*, I, Chap. I.

that the addition theorem in question is deduced as an example from the general Abelian theorem. But we are interested in seeing an elementary treatment of the curves of deficiency unity and the elliptic integrals, such as might follow or even form a part of a first course in the theory of functions, enriched and to some extent dominated by the general methods set forth in this work; and it seems to us an excellent feature of this treatise that the authors have elsewhere given so much attention to just this subject.

CHAPTER X is devoted to the Problem of Inversion and begins with the question: An irreducible algebraic equation $F(z, u) = 0$ being given, is it possible to find a rational function $R(z, u)$ such that, when

$$w = \int_{(z_0, u_0)}^{(z, u)} R(z, u)dz,$$

to each value of w corresponds one and only one point of the Riemann surface? And the question is answered in the most direct and simplest manner possible. There follows a proof that if the deficiency of F is 1 and w is the everywhere finite integral, the points of the F-surface correspond in a one-to-one manner to the points of a parallelogram in the w-plane. (Since conformal representation is not considered in this treatise, no reference is made to that property of the transformation.) This theorem is a familiar one, but proofs of it are rare in treatises in which the integral w (not the doubly periodic function) forms the point of departure. The close analogy that exists between the properties of rational functions on a Riemann surface of deficiency 1 and those of single-valued doubly periodic functions is emphasized, some of the most important corresponding theorems being collected and placed side by side (p. 450). The problem studied by Briot and Bouquet: when will the algebraic differential equation

$$F\left(u, \frac{du}{dz}\right) = 0$$

have a single-valued solution $u = f(z)$? is treated and applications are given, the most important being the ascertainment of all single-valued functions that have an algebraic addition theorem. p. 454, middle, for dt/dz read dz/dt. A generalization of Briot and Bouquet's problem follows. After this excellent introduction, which forms in fact a chapter in the theory of the elliptic integrals, the authors proceed to Jacobi's problem of inversion for the Abelian integrals.

CHAPTER XI: Normal curves, Moduli. The chapter opens with the theorem: "There exists no birational transformation, containing an arbitrary parameter, that transforms a curve of deficiency greater than unity into itself." The most general transformation of a curve of deficiency unity into itself is obtained in terms of the parameter w. Clebsch's and Noether's normal curves are studied and the number of moduli of the algebraic configuration is determined by counting off parameters. Thus a lower limit to the number of the moduli is rigorously established and an elegant application of this result is made to the proof of the theorem that an algebraic plane curve with non-specialized coefficients is not the most general curve of its deficiency.

CHAPTER XII presents some geometric applications of Abel's theorem, principally those to the groups of points of intersection of curves. The points of inflection of a cubic and the points where a conic has contact of the fifth order, the systems of conics tangent to the quartic wherever they meet it, the double-tangents of the quartic, and the discussion of some of these topics when the ground-curve has double-points, are among the subjects treated; but a few examples in areas, angles and lengths of arcs are taken up at the end.

The reviewer has expressed his mind freely on those points where his views differ from those of the authors. He begs leave to say in closing that his admiration for their work is none the less hearty because of these differences of opinion and to add that their treatise seems to him to be a thoroughly desirable book to put into the hands of the student.

W. F. OSGOOD.

HARVARD UNIVERSITY,
June 1896.

PAPERS OF THE MATHEMATICAL CONGRESS.

Mathematical Papers Read at the International Mathematical Congress held in Connection with the World's Columbian Exposition, Chicago, 1893. Edited by the Committee of the Congress: E. HASTINGS MOORE, OSKAR BOLZA, HEINRICH MASCHKE, HENRY S. WHITE. New York, The Macmillan Company, for the American Mathematical Society. 1896. 8vo. XVI. and 411 pp. $4.00.

THIS volume, which enjoys the distinction of being the first to be published in the form of a separate book by the

American Mathematical Society, has a number of claims to the attention of mathematical readers besides the intrinsic merits of the papers which it contains. One of these is the significance of the occasion which it commemorates. International Congresses bid fair to play an important rôle in the mathematical life of the future and the Chicago Congress is the initial gathering of the sort. Another is the highly international character of the collection itself, twenty-one of its papers having been contributed by German mathematicians, fourteen by American, four by French, three by Italian, two by Austrian and one by a Russian. And still another is the prominence in it of interesting *resumés* of recent progress in various departments of mathematical science, for the greater part admirably done by men who have themselves made important contributions to the departments of which they write. The success of the Congress itself was in large measure due to the presence and active participation of Professor Klein, of Göttingen. And the present volume owes to him not only the address made at the opening of the Congress on *The Present State of Mathematics*, but a number of papers by his university colleagues "which give collectively a fairly complete account of contemporaneous mathematical activity in Germany." Dr. Burkhardt, of Göttingen, discusses the recent developments in the *theory of perturbations*, examining the methods of Hill, Gylden and Linstedt from a mathematical point of view, particularly in their relation to the theory of the irregular integrals of linear differential equations. Dr. Fricke, now of Braunschweig, reviews the relations of the *theory of automorphic functions* to the higher arithmetic. Dr. Heffter, of Giessen, sketches the advances made in Germany in the theory of *linear differential equations* since the appearance of the epoch-making memoirs of Fuchs (in volumes 66 and 68 of *Crelle's Journal*). Professor Netto, of Giessen, expounds the thesis of *Kronecker* that it must ultimately be possible to "arithmetize" the whole of analysis, to reduce it to a body of relations expressible in finite terms among integral numbers, and his methods for accomplishing this reduction—the modular systems and the "association" with the algebraic functions of linear combinations of them with indeterminate coefficients.

Professor Schoenflies, of Göttingen, shows the significance of the theory of groups for modern *crystallography*. And Professor Study, now of Bonn, makes a review of "older and newer" investigation of systems of *complex numbers*, those especially in which these systems are considered in connection with transformation groups. And besides these reviews by German mathematicians there is an account by Professor Eddy, now of the University of Minnesota, of the remarkable development of *graphical methods* within the past thirty years, particularly in connection with engineering theory and practice.

The same general interest attaches to the paper in which Professor Hilbert, now of Göttingen, gives an outline of his own important investigations in the field of *algebraic invariants*; to that of Professor Minkowski, now of Königsberg, on his *geometry of integral numbers*; of Professor Pringsheim, of Munich, on his general theory of the *divergency and convergency of series* with positive terms; of Professor Study on his researches in *spherical trigonometry*; and of Mr. D'Ocagne, of Paris, on his powerful graphical method: "*Nomographie.*"

But the volume is not simply a collection of expository papers and *resumés*. It contains a full share of valuable and interesting original papers also. In the field of the *theory of substitutions* and *algebraic equations* are papers by Cole now of Columbia, Maschke and Moore of Chicago, de Perrott and Taber of Clarke and Weber of Strasburg; in the field of *functions*, papers by Bolza of Chicago, Hermite of Paris, Krause of Dresden, Pincherle of Bologna and Stringham of the University of California; while mathematical *history*, the reduction of *binary quantics*, *continuous transformation groups*, *algebraic curves*, *the Grassmann analysis* and *geodesic lines* are represented by papers by Halsted of the University of Texas, Hurwitz of Zürich, Meyer of Clausthal, Noether of Erlangen, Schlegel of Hagen and Weyr of Prague respectively. From every point of view, therefore, this volume is an important contribution to the mathematical literature of America. It is, moreover, carefully edited and beautifully printed.

HENRY B. FINE.

ELECTRICITY AND MAGNETISM.

Elements of the Mathematical Theory of Electricity and Magnetism. By J. J. THOMSON, F. R. S., Cavendish Professor of Experimental Physics in the University of Cambridge. vi + 510 pp., octavo. Cambridge, University Press; New York, The Macmillan Company, 1895. $2.60.

When Professor Thomson undertook to edit a new edition of Clark Maxwell's treatise it was hoped he would re-model parts of it in order to make clearer some conceptions upon which subsequent thought and experiment have thrown new light, and remove certain inconsistencies in Maxwell's method of presentation. It was felt that Professor Thomson was just the man to do this, and after consideration of the volume before us we are quite certain of it. Accordingly there was some disappointment when few changes were made, and the additions were collected in an extra volume, and not incorporated into the text. We presume that it was considered wiser not to tamper with Maxwell's classic treatise. When the publishers announced an elementary text-book on the mathematical theory of electricity and magnetism by J. J. Thomson, we were led to expect something new and individual in treatment, and embodying the latest work on the subject; this expectation has been fully realized. There was no dearth of text-books on this subject before, but they were, one and all, very unsatisfactory; although written presumably for physicists, the physical aspect of the subject was frequently not even considered. Perhaps the first impression after reading a few dozens of pages at the beginning of the volume before us, is that there is too close an adherence to Maxwell's matter and method ; but the individuality of the book is soon apparent, new methods of presentation please us, and difficult theorems yield before a simple treatment and careful choice of illustrative cases.

The first five chapters are devoted to the consideration of electro-statics. The author, at an early stage, introduces the use of Faraday tubes. Their value in the formation of mental pictures of the phenomena, not only of electro-statics, but of the whole subject, he conclusively shows us in an excellent chapter in his "Recent Researches." To the student of physics, in general, and of electricity, in particular, all such aids to thought are especially acceptable. The more use is made of the notion of potential, as work done, and of lines of force in the explanation of such phenomena as the variation of capacity, the attractions and repulsions of electrified bodies, the energy in the various parts of the field, etc., the more hold has the student upon the theorems considered. The last few sections of Chapter II. of this book, and again Article 74, afford an admirable example of the uses to which the conception of tubes of force may be put.

There is one point about the book which seems to us to spoil somewhat its excellence ; that is, a certain amount of what has been called "calculus dodging." It is hard to see why a theorem such, for example, as Gauss's, Art. 10, or the energy in the electric field, Art. 23, should be proved by such long round-about geometrical or other methods, rather than by the use of the shorter calculus; especially as calculus is used in all parts of the book.

The author states in his preface that very often important theorems are left out of elementary text-books, because their proof in their most general form demands higher analysis than can be required of the student, and contends "that often the study of the cases which will yield to very simple mathematical treatment fully illustrates the physical phenomena, and gives the student a more vivid idea and a more manageable grasp of the subject than he would be likely to attain, if he merely regarded electrical phenomena through a cloud of analytical symbols." A confirmation of this is his treatment of the law of the inverse square in attraction, Art. 20, and his method of finding the tensions and pressures in the ether. The latter is a case of other than calculus methods referred to above which is eminently proper. Again see Articles 231 and 239 on the distribution of currents in a conductor.

It is interesting to see the whole subject of dielectrics considered without the introduction of that unnecessary phrase "fictive layer." If we are to follow Maxwell's theory, what real difference is there between a charge and a "fictive layer" of electricity?

Chapters VI., VII. and VIII. are devoted to magnetism. The student of electrical engineering will regret that more space is not given to a discussion of magnetic circuits, reluctance, etc.

Then follow chapters on electric currents, electromagnetism, units, dielectric currents and a very short chapter on thermo-electric currents. A notable novelty is his treatment of the actual distribution of currents in a network of wires by introducing the dissipative function, that is, the expression for the rate of heat production. He shows that it is only necessary to write down this expression and make it a minimum, subject to the condition that the algebraical sum of the currents which meet in a point is zero, in order to find these currents. The discussion of electrolysis is exceedingly scanty.

In a volume that is so generally excellent, it is not easy to point out parts of special merit, but perhaps the chapters on specific inductive capacity and electro-magnetic induction are particularly noteworthy. The usefulness of the book would be increased greatly by more complete tables, and by copious references: in the latter point it is entirely deficient. The freedom from mistakes is worthy of note in a first edition, if we neglect the inaccuracies in the reference to articles. In the last line of p. 144 the last term should be $\frac{a}{f^2-a^2}$ not $\frac{ea}{f^2-a^2}$. In line 8 of p. 157 the last term should be $\frac{1}{2}\frac{b^2}{a^2}$ not $\frac{1}{2}\frac{b^2}{a^3}$.

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THE SUBSTITUTION GROUPS WHOSE ORDER IS THE PRODUCT OF TWO UNEQUAL PRIME NUMBERS.

BY DR. G. A. MILLER.

The regular groups of order pq (p and q represent prime numbers, $p > q$) were determined by Netto.* It is the object of this paper to determine *all* the substitution groups of this order and to find formulas by means of which their number may be readily found for any given values of p, q and n † even if the groups themselves are unknown.

We shall determine the cyclical and the noncyclical groups separately. It will be found that the number of the noncyclical groups, when the values of p and q are such that groups of this type occur, is always equal to or one larger than the number of the cyclical groups for any given values of p, q and n.

We shall employ the symbol $R\,G_{pq}^{mpq}$ to represent the intransitive group which is obtained by establishing a simple isomorphism between m regular groups of order pq. The symbol G_y^x will be used to represent, as usual, any group whose degree is x and whose order is y.

§ 1

The Cyclical Groups.

Each one of these groups belongs to one and only one of the following three classes:

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* Theory of Substitutions (Cole's edition) § 130.

† n represents the degree of the groups.

(1) Those which may be formed by establishing a $p, 1$ isomorphism between RG_{pq}^{mpq}* and G_q^{n-mpq}, $(m \geqq 1)$. Groups of this class can occur only when $w=a\,q$.† Their number is the largest integer which does not exceed $\frac{a}{p}$.

(2) Those which may be formed by establishing a $q, 1$ isomorphism between RG_{pq}^{mpq}, and G_p^{n-mpq}, $\left(m, \frac{n-mpq}{p} \geqq 1\right)$. Groups of this class can occur only when $n=bp$. Their number is the largest integer which is less than $\frac{b}{q}$.

(3) Those which may be formed by establishing a simple isomorphism between RG_{pq}^{mpq} and $G_p^{cp} \times G_q^{n-cp-mpq}$, $\left(c, \frac{n-cp-mpq}{q} \geqq 1\right)$. Their number is equal to the number of solutions of the following Diophantine equation

$$(\mathrm{A}') \qquad px+qy=a'+k'pq \ (k'=0, 1, 2, \cdots, k'_1)$$

$(x, y \geqq 1$; a'=smallest number‡ for which $px+qy=a'$ can be solved and which also satisfies the equation $a+k_1'pq=n$. The last equation determines also k_1'.) Since (A') has one solution for $k'=0$, two for $k'=1$, three for $k'=2$, etc., the number (N') of the groups of this class can be readily found by means of the following formula:

$$N'=\frac{1}{2}(k_1'+1)(k_1'+2)$$

The groups are found by making $m=k'$, $c=x$ and $\frac{n-cp+mpq}{q}=y$ in the given simple isomorphism; k', x and y being any set of values that satisfies (A'). All the possible cyclical groups are hereby determined. Their total number (N_1) is therefore equal to the number of solutions of

$$(\mathrm{A}) \qquad px+qy=a+kpq \ (k=0, 1, 2, \cdots, k_1)$$

---

* Only the cyclical groups represented by this symbol occur in this §.

† The number symbols throughout the article represent only positive integers.

‡ $a' \leqq 2pq$, cf. Jordan's Traité des substitutions, p. 4. Hence a' can always be found by at most two simple trials.

$(x, y \geqq 1$ when $k = k_1$; $a =$ the smallest number* for which $px + qy = a$ can be solved and which also satisfies the equation $a + k_1 pq = n$. The last equation determines also k_1.) Hence N_1 may be found directly by means of the following formula

$$N_1 = \frac{1}{2}(k_1+1)(k_1+2) - m_1$$

$m_1 = 2$ when n is an exact multiple of pq, $m_1 = 1$ when n is an exact multiple of either p or q but not of both, $m_1 = 0$ when n is an exact multiple neither of p nor of q.

As k_1 and m_1 can easily be determined for any given values of p, q and n the given formula is convenient. When $n = pq$ the corresponding group† will be transitive. All the other groups are intransitive.

§ 2

The Noncyclical Groups.

Each one of these groups contains one and only one self-conjugate sub-group. Its order is p. If non-regular transitive groups exist they must be of degree p‡ and sub-groups of the metacyclic group $G^p_{p(p-1)}$. This group has one and only one sub-group for each one of the different divisors of $p-1$ §. Hence there is one non-regular transitive group, as well as one regular noncyclical group of order pq whenever q is a divisor of $p-1$.

In the following we shall assume that this condition is fulfilled. We shall denote the group obtained by establishing a simple isomorphism between l of these non-regular transitive groups by the symbol H^{lp}_{pq}.

We may construct all the possible non-cyclical groups by first establishing a p, 1 isomorphism between H^{lp}_{pq} and $G_q^{n-mpq-lp}$, and then a simple isomorphism between this group and G^{mpq}_{pq} ‖. It is clear that m and l cannot both be zero at the same time. The number (N_2) of these groups is, therefore, equal to the number of the solutions of the following equation

$$(B) \qquad px + qy = a + kpq \quad (k = 0, 1, 2, \cdots k_1)$$

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* $a < 2pq$ cf. foot-note on a'.

† In this case $a = 0$, $k_1 = 1$ and $m_1 = 2$. Hence $n_1 = \frac{2.3}{2} - 2 = 1$.

‡ Since their order must be a multiple of their degree and they must contain the given self-conjugate sub-group.

§ Cf. Netto, l. c. § 132.

‖ Only the non-cyclical groups represented by this symbol occur in this §.

[$x \geqq 1$ when $k=k_1$; a and k satisfy the same conditions as in (A)]. Hence the formula

$$N_2=\tfrac{1}{2}(k_1+1)(k_1+2)-m_2$$

$m_2=1$ when n is an exact multiple of q, $m_2=0$ when this is not the case.

The groups may be constructed by making $m=k$, $l=x$ and $\frac{n-mpq-lp}{q}=y$ in the given isomorphism, where k, x and y are any set of values that satisfy (B).

The total number of groups of order pq is, therefore, equal to N_1 when q is not a divisor of $p-1$. In the other case it is equal to $N_1+N_2=N_3$, where

$$N_3=(k_1+1)(k_1+2)-m_3.$$

$m_3=3$ when n is an exact multiple of pq; $m_3=2$ when n is an exact multiple of q but not of p; $m_3=1$ when n is an exact multiple of p but not of q; $m_3=0$ when n is an exact multiple of neither p nor q.

§ 3

Examples.

We shall employ the last formula to determine the numbers of the groups of orders 6 and 10 when $n \leqq 10$* and when $n=50$. It is not difficult to construct the groups according to the given methods.

Order	*Degree*	a	N_3
6	3	3	$(0+1)(0+2)-1=1$
	4	4	$(0+1)(0+2)-2=0$
	5	5	$(0+1)(0+2)-0=2$
	6	0	$(1+1)(1+2)-3=3$
	7	7	$(0+1)(0+2)-0=2$
	8	2	$(1+1)(1+2)-2=4$
	9	3	$(1+1)(1+2)-1=5$
	10	4	$(1+1)(1+2)-2=4$
	50	2	$(8+1)(8+2)-2=88$
10	5	5	$(0+1)(0+2)-1=1$
	6	6	$(0+1)(0+2)-2=0$
	7	7	$(0+1)(0+2)-0=2$
	8	8	$(0+1)(0+2)-2=0$
	9	9	$(0+1)(0+2)-0=2$
	10	0	$(1+1)(1+2)-3=3$
	50	0	$(5+1)(5+2)-3=39$

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* These results may be compared with the lists of these groups in the *Quarterly Journal of Mathematics*, vols. 25, 26 and 27. For the groups of order 6 and degree 9 cf. footnote, ibid. vol. 27, p. 102.

It is clear that the number of these groups increases very rapidly with the increase of n. *E. g.*, when $n=1000$, there are $(166+1)(166+2)-2=28{,}054$ of order 6.

This article may be regarded as a continuation of "*The substitution groups whose order is four*," *Philosophical Magazine*, vol. 41 (1896), pp. 431–437.

PARIS, *May*, 1896.

NOTE ON THE SPECIAL LINEAR HOMOGENEOUS GROUP.

BY PROFESSOR HENRY TABER.

On page 232 of the BULLETIN are given the conditions necessary and sufficient in order that a transformation of the special linear homogeneous group in n variables may be generated by the repetition of an infinitesimal transformation of this group. As a corollary of these conditions it follows that a transformation of this group can be generated thus if the multiplicities of the several roots of its characteristic equation have no common factor, or if the roots of its characteristic equation are all equal to $+1$. But one or other of these conditions is satisfied if n is an odd prime. Therefore, if n is an odd prime, every transformation of the special linear homogeneous group in n variables can be generated by the repetition of an infinitesimal transformation of this group, that is, belongs to a continuous one-term sub-group containing the identical transformation.

On the other hand, if $n=2$ or is composite, it follows from the conditions given on page 232 that the special linear homogeneous group in n variables contains an assemblage of transformations no one of which can be generated by the repetition of an infinitesimal transformation of this group. Nevertheless, by the repetition of an infinitesimal transformation of this group we may approximate as nearly as we please to any transformation of this assemblage. Thus corresponding to any transformation A of the special linear homogeneous group that cannot be generated by the repetition of an infinitesimal transformation of this group can always be found a transformation A_ρ of this group, whose coefficients are rational functions of a parameter ρ, such that for all but a finite number of the values of ρ, A_ρ can be generated by the repetition of an infinitesimal transformation of this group, and by taking ρ sufficiently near to some constant value, as zero, A_ρ may be made as nearly as we please equal to A.

The method of demonstrating this theorem in general may be illustrated by one or two examples. Thus, for $n=2$, there is a doubly infinite system of transformations of the special linear homogeneous group no one of which can be generated by the repetition of an infinitesimal transformation of this group. For a proper choice of coördinates any transformation A of this system is defined by the equations

$$x'=-x+y, \qquad y'=-y.$$

Let A_ρ denote the transformation defined by the equations

$$x'=(\rho-1)x+y, \quad y'=\frac{1}{\rho-1}y;$$

A_ρ is then a transformation of the group, and passes continuously into the transformation A as ρ approaches zero continuously. For values of ρ sufficiently near to zero, the roots, $\rho-1$ and $(\rho-1)^{-1}$, of the characteristic equation of A_ρ are distinct; and therefore by a change of coördinates A_ρ will be defined by the equations

$$X'=(\rho-1)X, \qquad Y'=\frac{1}{\rho-1}Y,$$

where X and Y are the new coördinates of the point represented formerly by the coördinates x, y.*

Let $\rho-1=e^{\alpha+\beta\sqrt{-1}}$, where α is real and β is the smallest positive argument of $\rho-1$; and let B_ρ denote the transformation defined by the equations

$$X'=e^{\frac{1}{m}(\alpha+\beta\sqrt{-1})}X, \qquad Y'=e^{-\frac{1}{m}(\alpha+\beta\sqrt{-1})}Y$$

where m is any positive integer. B_ρ is then a transformation of the special linear homogeneous group, and B_ρ^m is equal to A_ρ. By taking m sufficiently great, B_ρ may be made as nearly as we please equal to the identical transformation. Therefore, by the repetition of an infinitesimal transformation of the special linear homogeneous group we may approximate as nearly as we please to A.

If in the m^{th} power of B_ρ we put $\rho=0$, we obtain the finite

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*The formulæ for the transformation of coördinates are

$$x=X-\frac{(\rho-1)}{\rho(\rho-2)}Y, \qquad y=Y.$$

transformation A; but the transformation B_ρ for $\rho=0$ is illusory, since for $\rho=0$ the formulæ for the transformation of coördinates are illusory. This is more readily apparent if we employ the original system of coördinates in the equations determining B_ρ, in which case B_ρ is defined by the equations

$$x'=e^{\frac{1}{m}(\alpha+\beta\sqrt{-1})}x+\frac{2(\rho-1)\sqrt{-1}}{\rho(\rho-2)}\sin\left(\frac{\alpha+\beta\sqrt{-1}}{m}\right)\cdot y,$$
$$y'=e^{-\frac{1}{m}(\alpha+\beta\sqrt{-1})}y.$$

If we regard m as varying continuously, then for any value of ρ other that 0, 1 or 2, the totality of transformations B_ρ constitute a continuous one-term group containing the identical transformation. The transformation A may be said to be a limit of this family of one-term groups, since, for a value of ρ sufficiently near to zero, certain of the transformations of the corresponding one-term group are as nearly equal as we please to A, which, nevertheless, does not belong to any one-term group of this family.*

For $n=6$, among the transformations of the special linear homogeneous group in n variables which cannot be generated by the repetition of an infinitesimal transformation of this group is the transformation A, defined by the equations

$$x_1'=\lambda x_1+x_2,\ x_2'=\lambda x_2+x_3,\ x_3'=\lambda x_3+x_4,\ x_4'=\lambda x_4$$
$$x_5'=\mu x_5+x_6,\ x_6'=\mu x_6,$$

where $\lambda=e^{\alpha+\beta\sqrt{-1}}$, $\mu=e^{-2\alpha+(\pi-2\beta)\sqrt{-1}}$

(α and β being both real). On the other hand, the transformation A_ρ defined by the equations

$$x_1'=\rho\lambda x_1+x_2,\ x_2'=\frac{\lambda}{\rho}x_2+x_3,\ x_3'=\lambda x_3+x_4,\ x_4'=\lambda x_4,$$
$$x_5'=\mu x_5+x_6,\ x_6'=\mu x_6,$$

which belongs to the special linear homogeneous group, and passes continuously into the transformation A as ρ approaches unity continuously, can, if λ and μ are fixed, for all but a finite number of the values of ρ, be generated by the repetition of an infinitesimal transformation of this group.

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* The transformation

$$x'=x+y,\quad y'=y$$

is also a limit of this family of one-term groups. It is obtained from B_ρ by putting $m=1$, and in the resulting transformation making ρ equal to 2.

Thus the roots of the characteristic equation of A_ρ are $\rho\lambda$, $\rho^{-1}\lambda$ each of multiplicity one and λ, μ each of multiplicity two; therefore, for all values of ρ which render $\rho\lambda$ distinct from the other roots of the characteristic equation; in particular if ρ is sufficiently near to unity, A_ρ belongs to a continuous one-term group containing the identical transformation.* By varying ρ we obtain a family of one-term group, to which, as before, A is a limit.

NOTES.

A REGULAR meeting of the American Mathematical Society was held in New York on Saturday afternoon, May 23, at three o'clock, the President, Dr. G. W. HILL, in the chair. There were thirteen members present. On the recommendation of the Council the following persons, previously nominated, were elected to membership: Mr. LOUIS TRENCHARD MORE, Johns Hopkins University, Baltimore, Md.; Professor HENRY ALLEN PECK, Syracuse University, Syracuse, N. Y. The following paper was read:

Dr. J. E. HILL: "Bibliography of surfaces and of twisted curves."

THE meeting of the British Association for the Advancement of Science will be held this year at Liverpool, opening on September 16. The President is Sir JOSEPH LISTER. Next year the Association will meet in Toronto.

THE German Mathematical Society will meet in conjunction with the German Association of Naturalists and Physicians at Frankfort-on-the-Main, September 21–26.

WE learn from *Nature* that the De Morgan Memorial Medal has this year been awarded by the Council of the London Mathematical Society to Mr. SAMUEL ROBERTS, F. R. S. This medal is awarded every three years. Mr. ROBERTS was one of the earliest members of the Society, and has published numerous papers, many of them of great value. The presentation will be made at the annual meeting of the Society in November next.

COLUMBIA UNIVERSITY. The Department of Mechanics will give the following graduate courses during the year 1896–97: By Professor R. S. WOODWARD: (1) Theory of the potential, with applications to gravitation, hydromechanics and geodynamics, two hours per week throughout the year; (2) Advanced analytical mechanics, embracing the theory of virtual displacements in statics and kinetics, the methods of Lagrange, Hamilton and Jacobi, with applications especially to rotating masses of variable form, two hours per week throughout the year. By Professor M. I. PUPIN: (3) Maxwell's theory of electricity and magnetism, two hours per week throughout the year; (4) Electromagnetic theory of light, three hours per week during the second half year. By Mr. J. C. PFISTER: (5) Theoretical mechanics (based on Ziwet's Theoretical Mechanics), two hours per week throughout the year.

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* See page 232.

NEW PUBLICATIONS.

I. HIGHER MATHEMATICS.

AUTONNE (L.). Sur la représentation des courbes gauches algébriques et sur le nombre des conditions qui expriment qu'une courbe algébrique est située sur une surface algébrique. Paris, Gauthier-Villars, 1896. 8vo. 41 pp.

BARKER (A. H.). Graphical calculus, with an introduction by J. Goodman. London, Longmans, 1896. 8vo. 198 pp. 4s. 6d.

BOREL (E.). See NIEWENGLOWSKI (B.).

BOYER (J.). Le mathématicien franc-comtois François-Joseph Servois, ancien conservateur du Musée d'artillerie, d'après des documents inédits (1767-1847). Paris, Gauthier-Villars, 1895. 8vo. 26 pp. Fr. 0.75.

BURALI-FORTI (C.). Il metodo del Grassmann nella geometria proiettiva. [*Rendiconti del Circolo matematico di Palermo*, Vol. 10, pp. 177-195.] 8vo.

CHEMNITZER (B.). Koordinaten-Transformationen auf beliebiger Oberfläche in konjugiert-komplexen Variablen. [Progr.] Annaberg, 1896. 4to. 32 pp.

DYCK (W.). Beiträge zur Potentialtheorie. I: Ueber die Darstellung der Kronecker'schen Charakteristik eines Functionensystems durch bestimmte Integrale. [*Sitzungsberichte der math.-phys. Classe der k. bayer. Akademie der Wiss.*, Vol. 25, pp. 261-277.] 8vo.

———. Beiträge zur Potentialtheorie. II: Die Gauss'sche Formel für die gegenseitige Umwindung zweier Raumcurven und ihre Ausdehnung auf höhere Mannigfaltigkeiten; Darstellung der Windungszahl zweier Mannigfaltigkeiten durch Kronecker'sche Charakteristiken gewisser Functionensysteme. [*ib.*, pp. 447-500.] 8vo.

GOLDSCHEIDER (F.). Ueber die Gauss'sche Osterformel. Teil I. [Progr.] Berlin, Gaertner, 1896. 4to. 30 pp. Mk. 1.00

Goodman (J.). See **Barker** (A. H.).

Grassmann (H.). Punktrechnung und projektive Geometrie. Teil II: Grundlagen der projektiven Geometrie. [Progr.] Halle, 1896. 4to. 28 pp.

Gutsche (O.). Neue Beweise und Ergänzungen zu Lehrsätzen Steiners über Kegelschnitte (Ges. Werke, Band 2, pp. 329–334, 345–347, 664–681). [Progr.] Breslau, 1896. 4to. 45 pp.

Gutzmer (A.). Zur Theorie der adjungirten Differentialgleichungen. [Habilitationsschrift.] Halle, 1896. 4to. 25 pp.

Halsted (G. B.). The essence of number. [*Texas Academy of Science*, 1896, pp. 61–63.] 8vo.

Hossfeld (C.). Beiträge zur Theorie der Raumkurven. [Progr.] Eisenach, 1896. 4to. 8 pp.

Jahrbuch über die Fortschritte der Mathematik, begründet von C. Ohrtmann; herausgegeben von E. Lampe. Vol. 25: Jahrgang 1893 und 1894. (In 3 Heften.) Heft 1. Berlin, Reimer, 1896. 8vo. 4 and 8 and 852 pp. Mk. 21.00

Kiechl (J.). Analytische Entwickelung von Gleichungen über zwei Transversalen eines Dreiecks. [Progr.] Feldkirch, 1895. 8vo. 12 pp., 1 plate.

Kotèlnikov (A. P.). The calculus of screws and some of its applications to geometry and mechanics. (In Russian.) Kazàn, University Press, 1895. 8vo. 4 and 216 pp.

Kusch (E.). C. G. J. Jacobi und Helmholtz auf dem Gymnasium. Beitrag zur Geschichte des Victoria-Gymnasiums zu Potsdam. [Progr.] Leipzig, Teubner, 1896. 8vo. 44 pp., 2 facsimiles. Mk. 1.60

Lampe (E.). See *Jahrbuch*.

Längst (H.). Kegelschnitte. Teil II: Analytisches Repetitorium und geometrischer Anhang, im Anschluss an den "Vorbereitenden Kurs" bearbeitet. Stuttgart, Kohlhammer, 1896. 8vo. 12 and 180 pp., 5 plates. Mk. 3.00

Loewy (A.). Ueber die Transformationen einer quadratischen Form in sich selbst, mit vorzüglicher Berücksichtigung der uneigentlichen, sowie ihre Anwendungen auf Linien- und Kugelgeometrie. [Diss.] München, 1895. 8vo. 66 pp.

Loos (W.). Analytische Behandlung einiger Grundprobleme der projektiven Geometrie. [Diss.] Giessen, 1896. 8vo. 31 pp., 1 plate.

Müller (W.). Transformationstheorie der Monge-Ampère'schen Differentialgleichungen. 1895. Leipzig, 1895. 8vo. 43 pp. Mk. 1.50

Niewenglowski (B.). Cours de géométrie analytique. (En 3 volumes.) Vol. III: Géométrie dans l'espace, avec une note sur les transformations en géométrie, par E. Borel. Paris, Gauthier-Villars, 1896. 8vo. 576 pp. Fr. 13.00

Picard (É.). Traité d'analyse. Vol. III.: Des singularités des intégrales des équations différentielles; étude du cas où la variable reste réelle; des courbes définies par des équations différentielles; équations linéaires; analogies entre les équations algébriques et les équations linéaires. (Cours de la Faculté des sciences de Paris.) Paris, Gauthier-Villars, 1896. 8vo. 14 and 568 pp. Fr. 18.00

Rougier (J.). Sur quelques sous-groupes de onzième classe du groupe modulaire. Marseille, Barthelet, 1896. 4to. 118 pp.

Seiffert (A.). Ueber eine neue geometrische Einführung in die Theorie der elliptischen Funktionen. [Progr.] Charlottenburg, 1896. 4to. 30 pp. 1 plate.

Suhle (H.). Zur Theorie der reellen Kurven einer rationalen Funktion *n*ten Grades für komplexe Variable. [Progr.) Dessau, 1896. 4to. 16 pp.

Suter (H.). Die Araber als Vermittler der Wissenschaften in deren Uebergang vom Orient in den Occident. Aarau, 1895. 8vo. 31 pp. Mk. 0.80

Tischer (E.). Ueber die Begründung der Infinitesimalrechnung durch Newton und Leibniz. [Progr.] Leipzig, Hinrichs, 1896. 4to. 46 pp. Mk. 1.00

II. ELEMENTARY MATHEMATICS.

August (E. F.). Vollständige logarithmische und trigonometrische Tafeln. 20ste Auflage. Leipzig, 1895. 12mo. 8 and 204 pp. Cloth. Mk. 1.60

Blakie (J.) and **Thomson** (W.). A text-book of geometrical deductions. Book I, corresponding to Euclid, book 1. New edition. Book II, corresponding to Euclid, book 2, with miscellaneous deductions from Books I and II. London, Thin, 1896. 8vo. 150 and 66 pp. 3s.

Blasendorff (M.). Ueber die Teilung des Kreisbogens. [Progr.] Berlin, Gaertner, 1896. 4to. 29 pp., 1 plate. Mk. 1.00

Bochow (K.). Eine einheitliche Theorie der regelmässigen Vielecke. Allgemeine Untersuchungen, nebst Berechnung der Seiten, Diagonalen und Flächen der im elementaren Unterrichte verwendbaren regelmässigen Vielecke aus den Reihen 2, 3, 5, 15, 17, 51, 85, 255. [Progr.] Leipzig, Fock, 1896. 4to. 34 pp., 2 plates, and 20 pp., 1 plate. Mk. 2.00

Bremiker (C). See **Crelle** (A. L.).

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Gauss (F. G.). Fünfstellige vollständige logarithmische und trigonometrische Tafeln. 48ste Auflage. Halle, 1896. 8vo. 166 and 35 pp. Half leather. Mk. 2.50

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Guglielmi (A.). Le nozioni di geometria solida, esposte per le scuole tecniche. 3a edizione, riveduta e corredata di nuovi esercizî. Napoli, Priore, 1896. 16mo. 52 pp. Fr. 0.80

Hue (T.) et **Vagnier** (N.). Géométrie (école de filles). Paris, Schmidt, 1896. 18mo. 196 pp.

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Manuel d'algèbre et de trigonométrie; par les Frères des écoles chrétiennes. Livre de l'élève. Paris, Poussielgue, 1896. 16mo. 8 and 216 pp.

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Martuscelli (E.). Complementi d'algebra; corso preparatorio allo studio del primo anno universitario (facoltà matematica) e per l'ammissione all'accademia militare. Salerno, Jovane, 1896. 8vo. 112 pp. Fr. 2.00

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Sickenberger (A.). Leitfaden der elementaren Mathematik. Teil 2: Planimetrie. 3te Auflage. München, Ackermann, 1896. 8vo. 6 and 123 pp. Mk. 1.50.

Spangenberg (E.). Arithmetic, algebra, plane and solid geometry explained to the practical tradesman. Vol. I. St. Louis, Laborer's Publishing Co., 1895. 32mo. 197 pp. $0.75

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Vade mecum geometrico per gli studenti del ginnasio, della scuola reale, per gli allievi delle accademie di marina e delle scuole nautiche, ossia raccolta ordinata di formole di planimetria, stereometria, trigonometria piana e sferica, di geometria analitica d'uso frequentissimo nella risoluzione di problemi ed assai utile per la ripetizione della teoria in preparazione di maturità. Trento, Manauni, 1896. 16mo. 33 pp. Fr. 0.60.

Vagnier (N.). See **Hue** (T.) et **Vagnier** (N.).

III. APPLIED MATHEMATICS.

Appell (P.). Traité de mécanique rationnelle. Vol. II: Dynamique des systèmes; mécanique analytique. (Cours de mécanique de la Faculté des sciences.) Paris, Gauthier-Villars, 1896. 8vo. 4 and 538 pp. Fr. 16.00.

Boulvin (J.). Cours de mécanique appliquée aux machines, professé à l'école spéciale du génie civil de Gand. 5e fascicule: machines à vapeur. Paris, Bernard, 1896. 8vo. 304 pp., 3 plates.

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GÉRARD (S.). Géométrie descriptive, à l'usage des candidats aux baccalauréats de l'enseignement secondaire moderne (2me partie, 2me série). Chateauroux, Majesté, 1896. 18mo. 33 pp.

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HERTZ (H.). Miscellaneous papers. With an introduction by P. Lenard. Authorized English translation by D. E. Jones and G. A. Schott. London and New York, Macmillan, 1896. 8vo. 26 and 340 pp. Cloth. $3.25

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HEYMANN (J.). Geometrisches Linearzeichnen. Als Vorbereitung auf das Projektionszeichnen an gewerblichen und technischen Schulen. Düsseldorf, Wolfram, 1896. 4to. 13 plates, 15 pp. text. Boards. Mk. 1.00

HOFF (J. H. VAN'T). See COHEN (E.).

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MEWES (R.). See CLAPEYRON (E.) and HUYGHENS (C.).

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Riem (J.). Ueber die Bahn des grossen Kometen 1881 III (Tebbut). [Diss.] Strassburg, 1895. 4to. 205 pp.

Roger (E.). Recherches sur le système du monde. 2e édition, entièrement refondue. Paris, Gauthier-Villars, 1896. 4to. Fr. 5.00

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Tisserand (F.). Traité de mécanique céleste. (En 4 volumes.) Vol. IV: Théories des satellites de Jupiter et de Saturne; perturbations des petites planètes. Paris, Gautnier-Villars, 1896. 4to. 12 and 548 pp. Fr. 28.00.

Weisbach (J.). Lehrbuch der Ingenieur- und Maschinen-Mechanik. Teil 3: Die Mechanik der Zwischen- und Arbeitsmaschinen. 2te Auflage, bearbeitet von G. Herrmann. 3te Abteilung, 1ste Hälfte: Die Maschinen zur Formveränderung. Braunschweig, Vieweg, 1896. 8vo. 10 and 1222 pp. Mk. 33.00.

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Züge (H.). Zum Problem der Anziehung homogener Ringkörper. [Progr.] Wilhelmshaven, 1896. 4to. 16 pp.

ERRATUM.

On page 228, line 8, of this volume, the words, "with contragredient variables into itself," should be replaced by "into itself, the further and nearer variables being transformed by linear substitutions transverse to each other."

FIFTH ANNUAL LIST OF PAPERS

READ BEFORE THE AMERICAN MATHEMATICAL SOCIETY AND SUBSEQUENTLY PUBLISHED, INCLUDING REFERENCES TO THE PLACES OF THEIR PUBLICATION.

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—— On the Calculus of Functions Derived from Limiting-Ratios. Read May 25 and Aug. 27, 1895. *Annals of Mathematics*, vol. 10, Nos 2 and 3, pp. 50–76; Jan. and March, 1896.

HATHAWAY, ARTHUR S. Elementary Proof of the Quaternion Associative Principle. Read Aug. 28, 1895. *Bulletin of the American Mathematical Society*, vol. 2, No. 2, pp. 43–45; Nov., 1895.

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The following dates have been fixed for the meetings of the Society:

	P. M.		P. M.
Sat., Oct. 31, 1896,	3:00	Sat., Feb. 27, 1897,	3:00
" Nov. 28, "	"	" Mar. 27, "	"
Wed., Dec. 30, "	"	" April 24, "	"
Sat., Jan. 30, 1897,	"	" May 29, "	"

The Summer Meeting will be held in Buffalo, N. Y., August 31 and September 1, 1896, in affiliation with the American Association for the Advancement of Science.

The meeting of Wednesday, December 30, 1896, will be the Annual Meeting of the Society for the election of Officers.

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